Example:

$$X_i \sim N(\mu, \sigma^2)$$

$P(U < c) = 0.9$?, $\quad U = \dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma'}$,

where $\sigma' = \sqrt{\dfrac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}{n-1}}$

$$\bar{X}_n \sim N(\mu, \sigma^2/n) \Longrightarrow \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$$

. by theorem 9 joint distr. of $\bar{X}_n$ and sample variance.

$$\sum_{i=1}^{n}\left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2 \sim \chi^2(n-1)$$

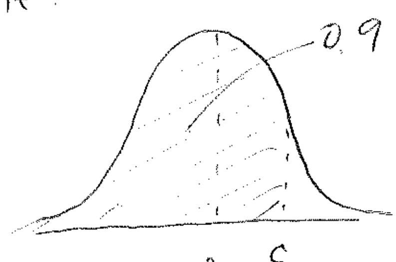
By definition of the t-distribution we know that if $Z \sim N(0,1)$ and $Y \sim \chi^2(m)$ and they are independent, then $U = \dfrac{Z}{\sqrt{\frac{Y}{m}}} \sim t_{(m)}$

So, considering $Z = \dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$ and $Y = \sum_{i=1}^{n}\left(\dfrac{X_i - \bar{X}_n}{\sigma}\right)^2$

we get that $\dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\dfrac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}{n-1}}} \sim t_{(n-1)}$.

So, back to the example:

$$U = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma'} \sim t_{(n-1)}$$



Now: $P(U < c) = 0.9$

c is the 0.9 quantile of a t distributed random variable with $n-1$ degrees of freedom.

from a table of the t distribution with $n-1 = 10-1 = 9$ degrees of freedom or using qt(0.9, df = 9) we get that $c = 1.383$.

What can we say about $\bar{X}_n - \mu$?

$P(U < 1.383) = 0.9$

$$P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma'} < 1.383 \right) = 0.9$$

$$P\left(\bar{X}_n - \mu < \frac{1.383\sigma'}{\sqrt{n}} \right) = 0.9$$

$$P\left(\bar{X}_n - \mu < 0.437\sigma' \right) = 0.9$$

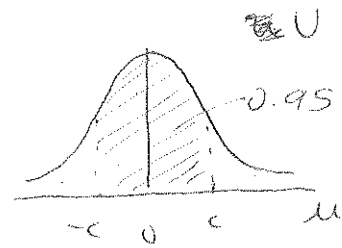
Confidence intervals

a) $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

b) $\overline{X_n}$ is an estimator of μ

$\sigma' = \sqrt{\dfrac{\sum_{i=1}^{n} (X_i - \overline{X_n})^2}{n-1}}$ is an estimator of σ^2.

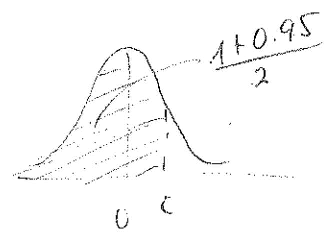
$U = \dfrac{\sqrt{n}\,(\overline{X_n} - \mu)}{\sigma'} \sim t_{(n-1)}$



c) $P(-c < U < c) = 0.95$

$$
\begin{aligned}
P(-c < U < c) &= P(U < c) - P(U < -c) \\
&= P(U < c) - P(U > c) \\
&= P(U < c) - [1 - P(U < c)] \\
&= 2\underbrace{P(U < c)}_{T_{n-1}(c)} - 1
\end{aligned}
$$

$$
\begin{aligned}
2T_{n-1}(c) - 1 &= 0.95 \\
T_{n-1}(c) &= \frac{1+0.95}{2} \\
c &= T_{n-1}^{-1}\left(\frac{1+0.95}{2}\right) !
\end{aligned}
$$



c is the $\frac{1+0.95}{2}$ quantile of a t distribution with $n-1$ degrees of freedom.

More generally, if we want to find c such that $P(-c < U < c) = \gamma$, then $\underbrace{c = T_{n-1}^{-1}\left(\frac{1+\gamma}{2}\right)}$, $0 < \gamma < 1$.

d) find ~~A and B such~~ A and B such that . $P(A < \mu < B) = 0.95$

From c)

$$P(-c < U < c) = 0.95 \quad , \quad c = T_{n-1}^{-1}\left(\frac{1+0.95}{2}\right)$$

$$U = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma'}$$

$$P\left(-c < \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma'} < c\right) = 0.95$$

$$P\left(-\frac{\sigma' \cdot c}{\sqrt{n}} < \bar{X}_n - \mu < \frac{\sigma' \cdot c}{\sqrt{n}}\right) = 0.95$$

$$P\left(-\bar{X}_n - \frac{c\sigma'}{\sqrt{n}} < -\mu < -\bar{X}_n + \frac{c\sigma'}{\sqrt{n}}\right) = 0.95$$

$$P\left(\bar{X}_n - \frac{c\sigma'}{\sqrt{n}} < \mu < \bar{X}_n + \frac{c\sigma'}{\sqrt{n}}\right) = 0.95$$

so $A = \bar{X}_n - \frac{c\sigma'}{\sqrt{n}}$, $c = T_{n-1}^{-1}\left(\frac{1+0.95}{2}\right)$

$B = \bar{X}_n + \frac{c\sigma'}{\sqrt{n}}$ $\quad \sigma' = \sqrt{\sum_{i=1}^{n}\left(\frac{x_i - \bar{X}_n}{\sigma}\right)^2}$

More general $P(A < \mu < B) = \gamma$, $0 < \gamma < 1$

where $A = \bar{X}_n - \frac{c\sigma'}{\sqrt{n}}$ $\quad c = T_{n-1}^{-1}\left(\frac{1+\gamma}{2}\right)$

$B = \bar{X}_n + \frac{c\sigma'}{\sqrt{n}}$, $\sigma' = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{X}_n)^2}{n-1}}$