Example: $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

- σ'^2 is an unbiased estimator for σ^2, that is, $E(\sigma'^2) = \sigma^2$

- $\hat{\sigma}^2$ is a biased estimator for σ^2, here $E(\hat{\sigma}^2) = \frac{(n-1)}{n}\sigma^2$

the bias of $\hat{\sigma}^2$ is $E(\hat{\sigma}^2) - \sigma^2 = \frac{(n-1)\sigma^2}{n} - \sigma^2 = \frac{-\sigma^2}{n}$.

$\hat{\sigma}_2^2 = g(\hat{\sigma}^2)$ such that $E(\hat{\sigma}_2^2) = \sigma^2$

is $\hat{\sigma}_2^2 = \frac{n}{n-1}\hat{\sigma}^2$ then $E(\hat{\sigma}_2^2) = E\left(\frac{n}{n-1}\hat{\sigma}^2\right) = \sigma^2$.

- $Var(\sigma'^2)$:

Recall that $\sum\left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2 \sim \chi^2_{(n-1)}$

$$E\left(\sum\left(\frac{X_i-\bar{X}_n}{\sigma}\right)^2\right) = n-1$$

$$Var\left(\sum\left(\frac{X_i-\bar{X}_n}{\sigma}\right)^2\right) = 2(n-1).$$

$$Var(\sigma'^2) = Var\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right) = Var\left(\frac{\sigma^2}{n-1}\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}_n}{\sigma}\right)^2\right)$$

$$= \frac{\sigma^4}{(n-1)^2}2(n-1) = \frac{2\sigma^4}{n-1}$$

- $Var(\hat{\sigma}^2) = Var\left(\frac{1}{n}\sum(X_i-\bar{X}_n)^2\right) = Var\left(\frac{\sigma^2}{n}\sum\left(\frac{X_i-\bar{X}_n}{\sigma}\right)^2\right) =$

$$= \frac{\sigma^4}{n^2}\cdot 2(n-1) = \frac{2(n-1)\sigma^4}{n^2}$$

M.S.E.

Example:

- Find the M.S.E. of σ'^2

the M.S.E. of σ'^2 is

$$E_{\sigma^2}\left[(\sigma'^2 - \sigma^2)^2\right] = Var_{\sigma^2}(\sigma'^2) + \left[E_{\sigma^2}(\sigma'^2) - \sigma^2\right]^2$$

$$= Var(\sigma'^2)$$ (because σ'^2 is an unbiased estimator for σ^2)

$$= \frac{2\sigma^4}{n-1}.$$

- Find the M.S.E. of $\hat{\sigma}^2$

$$E_{\sigma^2}\left[(\hat{\sigma}^2 - \sigma^2)^2\right] = Var_{\sigma^2}(\hat{\sigma}^2) + \left[E_{\sigma^2}(\hat{\sigma}^2) - \sigma^2\right]^2$$

$$= 2\frac{(n-1)\sigma^4}{n^2} + \left[-\frac{\sigma^2}{n}\right]^2$$

$$= \frac{2n\sigma^4 - 2\sigma^4}{n^2} + \frac{\sigma^4}{n^2}$$

$$= \frac{(2n-1)\sigma^4}{n^2}$$

$E(X) = \sum x \cdot P(X=x)$
$E(g(X))^{x} = \sum_{x} g(x)\, P(X=x)$

Limitations of unbiased estimators:

Example: $X \sim \text{Poisson}(\lambda)$. its p.f. $P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$, $x=0,1,2,\dots$

We are interested in estimating $P(X=0)^2$.

$P(X=0)^2 = \left[\frac{e^{-\lambda}\lambda^{0}}{0!}\right]^2 = [e^{-\lambda}]^2 = e^{-2\lambda}$. We are interested in estimating $e^{-2\lambda}$.

- one proposed estimator is $\delta(X) = (-1)^X$.

Recall that $\sum_{x=0}^{\infty} \frac{\alpha^x}{x!} = e^{\alpha}$

$$E_\lambda[\delta(X)] = E_\lambda[(-1)^X]$$
$$= \sum_{x=0}^{\infty} (-1)^x \cdot \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(-1\cdot\lambda)^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(-\lambda)^x}{x!}$$
$$= e^{-\lambda}\cdot e^{-\lambda} = e^{-2\lambda}$$

So $\delta(X) = (-1)^X$ is an unbiased estimator for $P(X=0)^2 = e^{-2\lambda}$

- other estimator for this problem is $\delta(X) = e^{-2X}$.

$$E_\lambda[\delta(X)] = E_\lambda[e^{-2X}] = \sum_{x=0}^{\infty} e^{-2x}\cdot\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^{-2}\cdot\lambda)^x}{x!}$$
$$= e^{-\lambda} e^{-\lambda e^{-2}}$$

$\delta(X) = e^{-2X}$ is a biased estimator for $P(X=0)^2 = e^{-2\lambda}$, but is much more appropriate ~~much~~ than $\delta(X) = (-1)^X$.

it can be shown that the M.S.E of $\delta(X) = e^{-2X}$ is smaller than the M.S.E. of $\delta(X) = (-1)^X$!!

Fisher Information:

~~Examples~~ ~~X ~ Berno~~

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2} \log f(X|\theta)\right]$$

Example: $X \sim$ Bernoulli (p).

$$f(x|p) = p^{x}(1-p)^{1-x}$$

$$\log f(x|p) = x \log p + (1-x)\log(1-p)$$

$$\frac{d}{dp} \log f(x|p) = \frac{x}{p} - \frac{(1-x)}{(1-p)}$$

$$\frac{d^2}{dp^2} \log f(x|p) = -\frac{x}{p^2} - \frac{(1-x)}{(1-p)^2}$$

$$I(p) = -E\left[-\frac{X}{p^2} - \frac{(1-X)}{(1-p)^2}\right]$$

$$= E\left[\frac{X}{p^2}\right] + E\left[\frac{1-X}{(1-p)^2}\right]$$

$$= \frac{p}{p^2} + \frac{(1-p)}{(1-p)^2}$$

$$= \frac{1}{p} + \frac{1}{1-p}$$

$$= \frac{1}{p(1-p)}$$

$X \sim$ Bernoulli(p)

$E(X) = p$.

$Var(X) = p(1-p)$