Classical and Bayesian inference

AMS 132

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Definition of an Unbiased Estimator

- Let $\boldsymbol{X} = (X_1, ..., X_n)$ be a random sample from a distribution that involves a parameter (or parameter vector) θ whose value is unknown.
- Suppose that we wish to estimate the parameter, θ, or a function of the parameter, $g(\theta)$.
- In a problem of this type, it is desirable to use an estimator $\delta(\boldsymbol{X})$ that, with high probability, will be close to θ or $g(\theta)$.

Example

Suppose that $\boldsymbol{X} = (X_1, ..., X_n)$ form a random sample from a normal distribution for which the mean θ is unknown and the variance is 1.

What is the M.L.E. of θ?
What does its distribution say about θ?

Definition of an Unbiased Estimator

Definition (Unbiased Estimator / Bias)

An estimator $\delta(\boldsymbol{X})$ is an *unbiased estimator* of a function $g(\theta)$ of the parameter θ if $E_\theta[\delta(\boldsymbol{X})] = g(\theta)$ for every possible value of θ.

An estimator that is not unbiased is called a *biased estimator*.

The difference between the expectation of an estimator and $g(\theta)$ is called the *bias* of the estimator. That is, the bias of δ as an estimator of $g(\theta)$ is $E_\theta[\delta(\boldsymbol{X})] - g(\theta)$, and δ is *unbiased* if and only if the bias is 0 for all θ.

Definition of an Unbiased Estimator

Example (Estimators for the variance in normal sampling)

Suppose that $\boldsymbol{X} = (X_1, ..., X_n)$ form a random sample from a normal distribution for which the mean θ is unknown and the variance σ^2 is unknown.

- Show that ${\sigma'}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$ is an unbiased estimator of σ^2.
- Show that $\widehat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$ is a biased estimator of σ^2. Find a new estimator, say $\widehat{\sigma}_2^2 = g(\widehat{\sigma}^2)$, that is unbiased for σ^2.
- Compute the variance of ${\sigma'}^2$ and $\widehat{\sigma}^2$.

Mean Squared Error

Definition (Mean Square Error)

Let δ be an estimator with finite variance. Then the mean square error (M.S.E.) of δ as an estimator of $g(\theta)$ is defined by

$$E_\theta\left[(\delta - g(\theta))^2\right].$$

- It can be shown that $E_\theta\left[(\delta - g(\theta))^2\right] = Var(\delta) + (E_\theta[\delta] - g(\theta))^2$.
- If δ is an unbiased estimator of $g(\theta)$, then the M.S.E. of δ is equal to its variance.
- Estimators with small M.S.E. are to be preferred.

Mean Squared Error

Example (Estimators for the variance in normal sampling)

Suppose that $\boldsymbol{X} = (X_1, ..., X_n)$ form a random sample from a normal distribution for which the mean θ is unknown and the variance σ^2 is unknown.

- Find the M.S.E. of ${\sigma'}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$.
- Find the M.S.E. of $\widehat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$.

Limitations of Unbiased Estimators

- An unbiased estimator might not exist.
- The unbiased estimator might be inappropriate.
- Information from the experiment might be ignored in the search of an unbiased estimator.

Example

Let X be a random variable from the Poisson distribution with mean λ, that describes the number of incoming calls at a telephone switchboard per minute. It is of interest to estimate the probability that no calls arrive in the next two minutes, this is, $P(X = 0)^2$.

- Show that $\delta(X) = (-1)^X$ in an unbiased estimator for $P(X = 0)^2$. Discuss how appropriate this estimator can be.
- Show that $\delta(X) = e^{-2X}$ is a biased estimator for $P(X = 0)^2$. Discuss how appropriate this estimator can be.

It can be shown that the estimator e^{-2X} has a smaller M.S.E. than $(-1)^X$!