$$\sum_{i=1}^{n} X_i \quad , \quad \sum_{i=1}^{n}(X_i - \mu) \quad , \quad \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 \quad , \quad |\bar{X}_n - \mu|$$

(+) we have assume that $X_1 \ldots X_n$ form a random sample from $N(\mu, \sigma^2)$

$$\bar{X}_n \sim N(\mu, \sigma^2/n)$$

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$$

$$\frac{n(\bar{X}_n - \mu)}{\sigma^2} \sim \chi^2_{(1)}$$

$$\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$

$$\frac{n\hat{\sigma}_0^2}{\sigma^2} \sim \chi^2_{(n)} \quad , \quad \hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

$$P(|\bar{X}_n - \mu| < 0.1) = P(-0.1 < \bar{X}_n - \mu < 0.1)$$
$$= P\Big(-\frac{\sqrt{n}}{\sigma}0.1 < \underbrace{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}}_{N(0,1)} < \frac{\sqrt{n}\,0.1}{\sigma}\Big)$$

$$P(\hat{\sigma}_0^2 < 0.1) = P\Big(\underbrace{\frac{\hat{\sigma}_0^2 \cdot n}{\sigma^2}}_{\chi^2_{(n)}} < \frac{n \cdot 0.1}{\sigma^2}\Big)$$

$$P(\hat{\sigma}_0^2 < \sigma^2) = P\Big(\underbrace{\frac{n\hat{\sigma}_0^2}{\sigma^2}}_{\chi^2_{(n)}} < n\Big)$$

Example: $X_i \sim N(\mu, \sigma^2)$, $i = 1, \dots, K$.

$\sigma^2 = 0.09$, $K = 10$, $\mu = 0.3$.

- $Y = \frac{1}{K} \sum_{i=1}^{n} |X_i - \mu|^2$; $P(Y \leq \mu^2)$

$X_i \sim N(\mu, \sigma^2)$, independent

$\frac{X_i - \mu}{\sigma} \sim N(0,1)$

$\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$

$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$.

$$P(Y \leq \mu^2) = P\left(\frac{1}{K} \sum_{i=1}^{K} |X_i - \mu|^2 \leq 0.3^2\right) = P\left(\sum_{i=1}^{K} |X_i - \mu|^2 \leq K \cdot 0.3^2\right)$$

$$= P\left(\underbrace{\sum_{i=1}^{K} \left(\frac{X_i - \mu}{\sigma}\right)^2} \leq \frac{K \cdot 0.3^2}{\sigma^2}\right)$$

$$= P\left(W \leq \frac{10 \cdot 0.3^2}{0.09}\right), \quad W \sim \chi^2(K)$$

$$= P(W \leq 10)$$

$$= 0.56$$

- $Y = |\bar{X}_n - \mu|$; $P(|\bar{X}_n - \mu| < 0.1)$

$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$.

$$P(|\bar{X}_n - \mu| < 0.1) = P(-0.1 < \bar{X}_n - \mu < 0.1) = P\left(-\frac{0.1\sqrt{n}}{\sigma} < \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} < \frac{\sqrt{n}\,0.1}{\sigma}\right)$$

$$= P\left(-\frac{0.1\sqrt{n}}{\sigma} < Z < \frac{0.1\sqrt{n}}{\sigma}\right), \quad Z \sim N(0,1).$$

$$= \Phi\left(\frac{0.1\sqrt{n}}{\sigma}\right) - \Phi\left(-\frac{0.1\sqrt{n}}{\sigma}\right) = \Phi\left(\frac{0.1\sqrt{10}}{\sqrt{0.09}}\right) - \Phi\left(-\frac{0.1\sqrt{10}}{\sqrt{0.09}}\right)$$

$X_i \sim N(\mu, \sigma^2)$

$\overline{X_n} \sim N(\mu, \sigma^2/n)$ $\qquad\qquad \overline{X_n} = \frac{1}{n}\sum X_i$

$$\sum_{i=1}^{n} \frac{(X_i - \overline{X_n})^2}{\sigma^2} \sim ?$$

notice that

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 = \sum_{i=1}^{n}\left(\frac{X_i - \overline{X_n} + \overline{X_n} - \mu}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n}\left(\frac{X_i-\overline{X_n}}{\sigma}\right)^2 + 2\sum_{i=1}^{n}\left(\frac{X_i-\overline{X_n}}{\sigma}\right)\left(\frac{\overline{X_n}-\mu}{\sigma}\right) + \sum_{i=1}^{n}\left(\frac{\overline{X_n}-\mu}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n}\left(\frac{X_i-\overline{X_n}}{\sigma}\right)^2 + 2\left(\frac{\overline{X_n}-\mu}{\sigma}\right)\left(\underbrace{\frac{\sum X_i - n\overline{X_n}}{\sigma}}_{0}\right) + n\left(\frac{\overline{X_n}-\mu}{\sigma}\right)^2$$

$$\underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2}_{\chi^2_{(n)}} = \underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\overline{X_n}}{\sigma}\right)^2}_{?} + \underbrace{n\left(\frac{\overline{X_n}-\mu}{\sigma}\right)^2}_{\chi^2_{(1)}} \qquad \left(\frac{\sqrt{n}(\overline{X_n}-\mu)}{\sigma}\right)^2$$

$$W = Y + Z, \quad Y \perp\!\!\!\perp Z \text{ by theorem}$$

$$E(e^{tW}) = E(e^{t(Y+Z)}) = E(e^{tY}\cdot e^{tZ}) = E(e^{tY})E(e^{tZ})$$

$$\left(\frac{1}{1-2t}\right)^{-\frac{n}{2}} = E(e^{tY})\cdot\left(\frac{1}{1-2t}\right)^{-\frac{1}{2}}$$

$$\Rightarrow E(e^{tY}) = \left(\frac{1}{1-2t}\right)^{-\frac{n}{2}}\left(\frac{1}{1-2t}\right)^{\frac{1}{2}} = \left(\frac{1}{1-2t}\right)^{-\frac{(n-1)}{2}}$$

$$\text{so } Y = \sum_{i=1}^{n}\left(\frac{X_i-\overline{X_n}}{\sigma}\right)^2 \sim \chi^2_{(n-1)}$$