$X_i \sim \chi^2_{(m_i)}$, $i = 1, \ldots, k$, X_i are independent

$$Y = \sum_{i=1}^{k} X_i \sim ??$$

We will find the distribution of Y using m.g.f.

$$E(e^{tY}) = E\left(e^{t\sum_{i=1}^{k} X_i}\right) = E\left(e^{t(X_1 + \cdots + X_k)}\right)$$

$$= E\left(e^{tX_1 + tX_2 + \cdots + tX_k}\right) = E\left(e^{tX_1} e^{tX_2} \cdots e^{tX_k}\right)$$

$$\overset{\text{Independence}}{=} E(e^{tX_1})\, E(e^{tX_2}) \cdots E(e^{tX_k})$$

$$= \left(\frac{1}{1-2t}\right)^{m_1/2} \left(\frac{1}{1-2t}\right)^{m_2/2} \cdots \left(\frac{1}{1-2t}\right)^{m_k/2}$$

$$= \left(\frac{1}{1-2t}\right)^{\frac{(m_1 + m_2 + \cdots + m_k)}{2}}$$

$\sim \chi^2_{(m)}$

$\left(\frac{1}{1-2t}\right)^{m/2}$

So we conclude that

$$\sum_{i=1}^{k} X_i \sim \chi^2_{(m_1 + \cdots + m_k)}.$$

this is the sum of <u>independently</u> distributed ~~χ^2 distributed~~ random variables with χ^2 distribution has a χ^2 distribution.

$$F_Y(y) = P(Y \le y)$$
$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$X \sim N(0,1)$, $Y = X^2 \sim ??$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$$
$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

then the density function of $Y = X^2$ is $\boxed{f_Y(y) = \frac{d}{dy} F_Y(y)}$

$$f_Y(y) = \frac{d}{dy}\left[\Phi(\sqrt{y}) - \Phi(-\sqrt{y})\right]$$
$$= \phi(\sqrt{y})\frac{1}{2\sqrt{y}} + \phi(-\sqrt{y})\frac{1}{2\sqrt{y}}$$

ϕ is the density function of X

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\cdot y\right\}\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\cdot y\right\}\frac{1}{2\sqrt{y}}$$
$$= \frac{1}{\sqrt{2}\sqrt{\pi}} \frac{1}{\sqrt{y}} \exp\left\{-\frac{y}{2}\right\}$$
$$= \frac{1}{2^{1/2}\Gamma(\frac{1}{2})} \cdot y^{-\frac{1}{2}} e^{-\frac{y}{2}} \quad , y > 0$$

So, $Y = X^2 \sim \chi^2_{(1)}$ //

$X_i \overset{iid}{\sim} N(0,1), i=1..m$, $Y = \sum_{i=1}^{m} X_i^2 \sim ?$

we know that if $X \sim N(0,1) \Rightarrow X^2 \sim \chi^2(1)$
we know that if $Y_i \sim \chi^2(m_i) \Rightarrow \sum_{i=1}^{m} Y_i \sim \chi^2_{(m_1+\dots+m_m)}$

we have that $X_i^2 \sim \chi^2(1)$, since X_i's are independent.
it follows that $\sum_{i=1}^{m} X_i^2 \sim \chi^2(m)$.

Example

a) $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$.

$$\frac{n\hat{\sigma}_0^2}{\sigma^2} \sim ?$$

$$\frac{n\hat{\sigma}_0^2}{\sigma^2} = \frac{\cancel{n}\cdot\frac{1}{\cancel{n}}\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$$

$$= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$$

since $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\Rightarrow \frac{X_i-\mu}{\sigma} \sim N(0,1)$$

$$\Rightarrow \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$$

$$\Rightarrow \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$

b) show that $\hat{\sigma}_0^2 \sim Gama\left(\frac{n}{2}, \frac{n}{2\sigma^2}\right)$

we know tha $Y = \frac{n\hat{\sigma}_0^2}{\sigma^2} \sim \chi^2(n)$

$\cancel{X2}$ $X \sim \chi^2(n)$

$\hat{\sigma}_0^2 = \frac{\sigma^2}{n}X \sim ?$

$$F_Y(y) = P\left(\underbrace{\frac{n\hat{\sigma}_0^2}{\sigma^2}}_{\chi^2_{(n)}} \le y\right) = P\left(\hat{\sigma}_0^2 \le \frac{\sigma^2}{n}y\right)$$

$$P\left(\hat{\sigma}_0^2 \le y\right) = P\left(\underbrace{\frac{n\hat{\sigma}_0^2}{\sigma^2}}_{\chi^2_{(n)}} \le \frac{n}{\sigma^2}y\right)$$

$$Y = \frac{n\hat{\sigma}_0^2}{\sigma^2} \sim \chi^2_{(n)}$$

$\hat{\sigma}_0^2 = \frac{\sigma^2}{n}Y \sim ?$, $Y \sim \chi^2_{(n)}$

$$F_{\hat{\sigma}_0^2} = P\left(\frac{\sigma^2}{n} Y \le y\right) = P\left(Y \le \frac{n}{\sigma^2} y\right)$$

$$=$$

$$f_{\hat{\sigma}_0^2} = \frac{d}{dy} P\left(Y \le \frac{n}{\sigma^2} y\right)$$

$$= f_Y\left(\frac{n}{\sigma^2} y\right) \cdot \frac{n}{\sigma^2}$$

$$= \frac{n}{\sigma^2} \cdot \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \left(\frac{n}{\sigma^2} y\right)^{\frac{n}{2}-1} e^{-\frac{ny}{2\sigma^2}}$$

$$= \left(\frac{n}{\sigma^2}\right)^{\frac{n}{2}} \cdot \frac{1}{2^{n/2}} \cdot \frac{1}{\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{n}{2\sigma^2} \cdot y}$$

$$= \left(\frac{n}{2\sigma^2}\right)^{n/2} \frac{1}{\Gamma(n/2)} y^{\frac{n}{2}-1} e^{-\frac{n}{2\sigma^2} \cdot y}.$$

So $\hat{\sigma}_0^2 \sim \text{Gama}\left(\frac{n}{2}, \frac{n}{2\sigma^2}\right)$