Classical and Bayesian inference

AMS 132

February 9, 2018

Definition of the Distributions

- We will introduce and discuss a particular class of gamma distributions, known as chi-square (χ^2) distributions.
- These are very important in statistical inference.

Example (The M.L.E. of the Variance of a Normal Distribution)

Suppose that $X_1, \ldots, X_n$ for a random sample from the normal distribution with known mean μ and unknown variance σ^2. The M.L.E. of σ^2 is given by

$$\widehat{\sigma_0^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2.$$

- The distribution of $\widehat{\sigma_0^2}$ and $\widehat{\sigma_0^2}/\sigma^2$ are useful in several statistical problems.

Definition of the Distributions

Definition (χ^2 Distribution)

For each positive number m, the gamma distribution with parameters $\alpha = m/2$ and $\beta = 1/2$ is called the χ^2 distribution with m degrees of freedom and its p.d.f. is given by

$$f(x) = \frac{1}{2^{m/2}\Gamma(m/2)} x^{m/2-1} e^{-x/2}, \qquad x > 0$$

.

- We are going to denote this distribution as χ^2_m.
- It is common to restrict the degrees of freedom m to be an integer, but it is not necessary.
- Look plots in R.
- The χ^2 distribution with $m = 2$ is the exponential distribution with parameter $1/2$.
- The following distributions are the same: the gamma distribution with parameters $\alpha = 1$ and $\beta = 1/2$, the χ^2 distribution with two degrees of freedom, and the exponential distribution for which the mean is 2.
- By definition, $\Gamma(1/2) = \sqrt{\pi}$.

Properties of the Distributions

Theorem (Mean and Varaince)

If a random variable X has the χ^2 distribution with m degrees of freedom, then $E(X) = m$ and $Var(X) = 2m$.

- The moment generating function (m.g.f.) of X is

$$\Psi(t) = \left(\frac{1}{1-2t}\right)^{m/2}, \quad t < 1/2.$$

Theorem (Mean and Varaince)

If the random variables $X_1, \ldots, X_k$ are independent and if X_i has the χ^2 distribution with m_i degrees of freedom ($i = 1, \ldots, k$), then the sum $X_1 + \ldots + X_k$ has the χ^2 distribution with $m_1 + \ldots + m_k$ degrees of freedom.

Properties of the Distributions

Theorem (Relationship between $N(0,1)$ and χ^2)

Let X have the standard normal distribution, i.e. $X \sim N(0,1)$. Then the random variable $Y = X^2$ has the χ^2 distribution with one degree of freedom.

Proof:

Corollary

If the random variables $X_1, \ldots, X_m$ are i.i.d. with the standard normal distribution, then the sum of squares $X_1^2, \ldots, X_m^2$ has the χ^2 distribution with m degrees of freedom.

Proof:

Example (Distribution of the M.L.E. of the Variance of a Normal Distribution)

Suppose that $X_1, \dots, X_n$ for a random sample from the normal distribution with known mean μ and unknown variance σ^2. The M.L.E. of σ^2 is given by

$$\widehat{\sigma_0^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2.$$

- Find the distribution of $\frac{n\widehat{\sigma_0^2}}{\sigma^2}$.
- Show that $\widehat{\sigma_0^2} \sim Gamma(n/2, n/(2\sigma^2))$.