University of California, Santa Cruz
Department of Applied Mathematics and Statistics
Baskin School of Engineering
Classical and Bayesian Inference - AMS 132

Final Exam: Take home

Name:

Instructions: You can use R or a table to get any quantile or probability you need, but be clear of what distribution are you considering, what are you computing (a quantile or a probability), how many degrees of freedom, etc.
Take home part is due on Thursday, March 22, at 12:00 pm
Good luck!.

Consider n patients that are waiting for a kidney transplant. After a suitable match donor is found, patients will be transplanted one at the time. Assume that donors are randomly selected and they are suitable for transplant with probability θ. Let $X_1, \ldots, X_n$ form a random sample, where each X_i describes the number of donors that are tested before one suitable match for patient i is found, $i = 1, ..., n$. Assume that $X_1, \ldots, X_n$ have a geometric distribution with unknown probability θ, $0 < \theta < 1$, this is, $f(x \mid \theta) = (1-\theta)^{x-1}\theta$, $x = 1, 2, ...$ If a random variable has a geometric distribution, then its mean is $\frac{1}{\theta}$ and its variance is $\frac{1-\theta}{\theta^2}$.

From a random sample of size 112 patients, it was observed that the total number of donors that were tested, before all of them got a kidney transplant, was 1243, this is, $\sum_{i=1}^{112} x_i = 1243$.

1. Find the maximum likelihood estimator of θ. **[5 pts]**

2. Find the Fisher information, $I_n(\theta)$, contained in the random sample $X_1, \ldots, X_n$. **[5 pts]**

3. Obtain and compute a one-sided 95% confidence interval for θ based on the asymptotic distribution of the maximum likelihood estimator of θ, such that $P(\theta < B) = 0.95$, where B is a random upper bound.

   For this, consider that $P(V > G^{-1}(1-\gamma)) = \gamma$, where γ is the coefficient of the confidence interval, V is a pivot, and $G^{-1}(1-\gamma)$ is the $1-\gamma$ quantile of the pivot's distribution. **[5 pts]**

4. Assume that θ is now random and has a beta prior distribution with parameters $a = 2$ and $b = 15$. Find the posterior distribution of θ. **[5 pts]**

5. Find a 95% one-sided credible region for θ, such that $P(\theta < c \mid \boldsymbol{x}) = 0.95$.
   To find c, you can use the `R` function `qbeta(p, shape1, shape2)` that returns the p-th quantile of a beta distribution with parameters `shape1` and `shape2`. **[5 pts]**

6. One of the doctors in the hospital beliefs that at least 20% of donors are a match for a patient waiting for a transplant. Based on the observed data and using your answers in questions 3 and 5, would you agree with this doctor? Make explicit interpretations of the results in both questions. **[5 pts]**

Name:

- Bernoulli distribution: if X is a random variable with Bernoulli distribution with parameter $0 \leq \theta \leq 1$, then $f(x \mid \theta) = \theta^x (1-\theta)^{1-x}$, $x \in \{0, 1\}$, $E(X) = \theta$, and $Var(X) = \theta(1-\theta)$.

- Poisson distribution: if X is a random variable with Poisson distribution with parameter $\theta > 0$, then $f(x \mid \theta) = \frac{e^{-\theta}\theta^x}{x!}$, $x > 0$, $E(X) = \theta$, and $Var(X) = \theta$.

- Exponential distribution: if X is a random variable with exponential distribution with parameter $\theta > 0$, then $f(x \mid \theta) = \theta e^{-\theta x}$, $x > 0$, $E(X) = \frac{1}{\theta}$, $Var(X) = \frac{1}{\theta^2}$.

- Gamma distribution: if X is a random variable with gamma distribution with parameters $a > 0$, and $b > 0$, then $f(x \mid a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$, $x > 0$, $E(X) = \frac{a}{b}$, $Var(X) = \frac{a}{b^2}$.