Classical and Bayesian inference

AMS 132

Probability and Statistical Models

Motivating ideas

- AMS 131:
  "Suppose that the random variable X, describing the lifetime of an electronic component, has an exponential distribution with parameter $\theta = 0.5$ years. Compute the probability of the electronic component lasting less than 2 years."

- AMS 132:
  "A company sells electronic components and they are interested in knowing as much as they can about how long each component is likely to last. They can collect data on components that have been used under typical conditions"
  - What variable is of interest in this problem?
  - What distribution can we consider for describing this variable?
  - How can we use this data to learn about the unknown quantities?
  - What if the unknown quantities are random?
  - What is the average of the next 3 lifetimes that they can observe?

Probability and Statistical Models

Definition (Statistical model)

A *statistical model* consist of an identification of random variables of interest, a specification of a family of possible distributions for the random variables, the identification of any unknown parameters of those distributions, and (if desired) a specification for a distribution for the unknown parameters.

In general, a statistical model takes the form

$$\mathscr{F} = \{f(x \mid \theta) : \theta \in \Omega\}.$$

Probability and Statistical Models

Definition (Statistical inference)

A *statistical inference* is a procedure that produces a probabilistic statement about some or all parts of a statistical model.

Example (Examples of statistical inferences)

Suppose that X_1, X_2, X_3, X_4, X_5, are i.i.d. random variables describing the lifetime of 5 components.

- Produce a random variable (r.v.) Y (a function of $X_1, \ldots, X_5$) such that $P(Y > \theta \mid \theta) > 0.9$.
- Produce a r.v. that we expect to be close to θ.
- Compute how likely it is that the average of the next 3 lifetimes, $\frac{1}{3}\sum_{i=6}^{8} X_i$, is at least 2.
- Say something about how confident we are that $\theta < 0.4$ after observing $X_1, \ldots, X_5$.

Probability and Statistical Models

Definition (Parameter and Parameter space)

In statistical inference a characteristic or combination of characteristics that determine the joint distribution of the random variables of interest is called *parameter* of the distribution.
The set Ω of all possible values of the parameter (or vector of parameters), θ, is called *parameter space*.

Example

- *exp*(θ)
- *Normal*(μ, σ^2)
- *Binomial*(n, p)

- Ω has to include all possible values of θ!
- In most problems, there is a natural interpretation for the parameter as a feature of the possible distributions of our data.

Example of Statistical Inference

Example (Different kinds of inferences)

Suppose that $X_1, X_2, \ldots, X_8$ are i.i.d. r.v.s that describe daily temperatures (in Celsius degree) in a city during Winter.
Values `1.11, 2.87, 6.16, 2.48, 1.32, 4.26, 0.04, 2.00` were observed.

a) Describe a statistical model.

b) What is the probability of the mean temperatures being less than 1.6 degrees?

c) Choose a "good" representation for the unknown mean daily temperatures. Assuming a known variability for the temperatures equal to 4. (See `R`)

d) Assume that the mean daily temperature is a random variable and propose a distribution. (See `R`)

e) Update the belief in the random mean temperature using the data. (See `R`)

f) Compute the probability of the daily mean temperatures being between 1.26 and 3.21 before and after observing the temperatures. (See `R`)

General Classes of Inference Problems

- Prediction: to predict random variables that have not yet been observed. When the unobserved quantity to be predicted is a parameter, prediction is usually called *estimation*.
- Statistical Decision Problems: after the experimental data have been analyzed, we must choose a decision from some available class of decisions with the property that the consequences of each available decision depend on the unknown value of some parameter.
- Experimental Design: control over the type or the amount of experimental data that will be collected. Experimental design and statistical inference are closely related.
- Other Inferences: the range of possible models, inferences, and methods that can arise when data are observed in real research problems far exceeds what we can introduce here. Get an appreciation for what needs to be done when a more challenging statistical problem arises.

Definition of a Statistic

- Each statistical inference that we will learn how to perform in this class will be based on one or a few summaries of the available data.

Definition (Statistic)

Suppose that the random variables of interest are $X_1, \ldots, X_n$. Let r be an arbitrary real-valued function of n real variables. Then the random variable $T = r(X_1, \ldots, X_n)$ is called a *statistic*.

Example

- In the daily temperatures example, we can be interested in $T = \left|\frac{1}{8}\sum_{i=1}^{8} X_i - 2\right|$.