$X_i \sim \text{Poisson}(\theta)$ , i.i.d. because we have a random sample.

$n = 74$ , $\sum_{i=1}^{74} x_i = 192$

a) statistical model.

$X$ is the r.v. that describes the number of problems that a student solves.

$$f(x_1 .. x_n | \theta) = f_n(\underline{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!}$$

$$= \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!} , \quad \theta > 0. \quad , x_i \in \{0,1,2,\dots\}$$

$\Omega = \mathbb{R}^+$

b) a statistical inference problem is finding the estimate of the mean.

c) find the MLE of interest : mean and coef. of variation of the # of problems that students solve.

$$E(X) = \theta$$

$$CV(X) = \frac{\sqrt{Var(X)}}{|E(X)|} = \frac{\sqrt{\theta}}{\theta} = \frac{1}{\sqrt{\theta}}$$

i) Lets find the MLE of $\theta$.

$$f_n(\underline{x}|\theta) = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$$

$$L(\theta) = \log f_n(\underline{x}|\theta)$$

$\boxed{\log = \ln}$

$$= \log\left( \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod x_i!} \right)$$

$$= -n\theta + \cancel{\sum_{i=1}^{n}} \sum_{i=1}^{n} x_i \log(\theta) - \log \prod_{i=1}^{n} x_i!$$

$$\frac{d}{d\theta} L(\theta) = -n + \frac{\sum_{i=1}^{n} x_i}{\theta} = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{\theta} = n \qquad \frac{d}{d\theta}\left[-n + \frac{\sum x_i}{\theta}\right] = -\frac{\sum x_i}{\theta^2}$$

$$\Rightarrow \boxed{\theta = \frac{\sum x_i}{n}}$$

$$\frac{d^2}{d\theta^2} L(\theta) = -\frac{\sum x_i}{\theta^2} < 0 \quad \forall x_i, \theta$$

checking that $\theta = \frac{\sum x_i}{n}$ is a maximum

Therefore $\hat{\theta} = \frac{\sum X_i}{n}$ is the M.L. estimator

$\hat{\theta} = \frac{\sum x_i}{n}$ is the M.L. estimate.

$$\hat{\theta} = \frac{\sum x_i}{n} = \frac{192}{74}$$

Answer: the MLE of the mean of the # of problems that students solve is $\frac{192}{74} = 2.59$

Then the MLE for $CV(X)$ is $\frac{1}{\sqrt{\frac{\sum X_i}{n}}}$ (estimator)

the maximum likelihood estimate of $CV(\underline{x})$ is

$$\frac{1}{\sqrt{\frac{\sum x_i}{n}}} = \frac{1}{\sqrt{\frac{192}{74}}} = \frac{1}{\sqrt{2.59}}$$

d) $\theta$ is r.v. and ~~$\theta \sim$ Gamma~~ $\theta$ has a gamma distribution with mean and variance equal to 2.

$$\theta \sim \text{Gama}(a, b)$$

$$E(\theta) = \frac{a}{b} = 2 \qquad Var(\theta) = \frac{a}{b^2} = 2$$

$$\frac{1}{b} \cdot 2 = 2 \Rightarrow \boxed{\begin{matrix} b = 1 \\ a = 2 \end{matrix}}$$

so $\theta \sim \text{Gama}(2, 1)$

under square error loss function
Bayes estimate of $E(X)$ is $E(\theta \mid \underline{x})$
Bayes estimate of $CV(X)$ is $E\left(\frac{1}{\sqrt{\theta}} \mid \underline{x}\right)$

- Lets find the posterior distr. of $\theta$ given data.

$$\xi(\theta \mid \underline{x}) \propto f_n(\underline{x} \mid \theta)\, \xi(\theta)$$

$$= \underbrace{\frac{e^{-n\theta}\, \theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}}_{\text{likelihood}} \cdot \underbrace{\frac{1^2}{\Gamma(2)}\, \theta^{2-1} e^{-1 \cdot \theta}}_{\text{Gama distr.}}$$

$$\propto \underbrace{\theta^{2 + \sum x_i - 1}\, e^{-(n+1)\theta}}_{\text{looks like Gama}(\underbrace{2 + \sum x_i}_{194},\ \underbrace{n+1}_{75})}$$

so,
Bayes estimate for $E(X)$ is $\dfrac{2 + \sum x_i}{n+1} = \dfrac{2 + 192}{74 + 1} = \dfrac{194}{75}$

Bayes estimate for $CV(X)$ is

$$E\left(\frac{1}{\sqrt{\theta}} \,\Big|\, \underline{x}\right) = \int_0^\infty \frac{1}{\sqrt{\theta}}\, \xi(\theta \mid \underline{x})\, d\theta$$

$$= \int_0^\infty \theta^{-1/2} \frac{(75)^{194}}{\Gamma(194)} \theta^{194-1} e^{-75\theta}\, d\theta$$

$$= \frac{(75)^{194}}{\Gamma(194)} \underbrace{\int_0^\infty \theta^{194-1/2-1} e^{-75\theta}\, d\theta}_{\text{looks like Gamma}(194-1/2,\ 75)}$$

$$\vdots$$

$$= \frac{(75)^{1/2}\, \Gamma(193.5)}{\Gamma(194)}$$

$$= \frac{(75)^{194}}{\Gamma(194)} \int_0^\infty \frac{\Gamma(194-1/2)}{(75)^{194-1/2}} \underbrace{\frac{(75)^{194-1/2}}{\Gamma(194-1/2)} \theta^{194-\frac{1}{2}-1} e^{-75\theta}\, d\theta}_{\text{integrates to 1}}$$

$$= \frac{(75)^{194}}{\Gamma(194)} \frac{\Gamma(194-1/2)}{(75)^{194-1/2}}$$

$$= \frac{75^{1/2}}{\Gamma(194)} \Gamma(194-1/2)$$

e) distr. of a new observation given $X_1 = x_1 \ldots X_n = x_n$

$$f(x_{n+1} \mid x_1 \ldots x_n) = \int_0^\infty f(x_{n+1} \mid \theta)\, \underbrace{\xi(\theta \mid \underline{x})}_{\sim \text{Gamma}(193.5,\, 75)}\, d\theta$$

$$= \int_0^\infty e^{-\theta} \frac{\theta^{x_{n+1}}}{x_{n+1}!} \frac{(75)^{193.5}}{\Gamma(193.5)} \theta^{193.5-1} e^{-75\theta}\, d\theta$$

$$= \frac{(75)^{193.5}}{\Gamma(193.5)} \frac{1}{x_{n+1}!} \int_0^\infty \theta^{193.5-1} e^{-76\theta}\, d\theta$$

$$= \frac{(75)^{193.5}}{\cancel{\Gamma(193.5)}} \frac{1}{x_{n+1}!} \frac{\cancel{\Gamma(193.5)}}{(76)^{193.5}} \underbrace{\int_0^\infty \frac{(76)^{193.5}}{\Gamma(193.5)} \theta^{193.5-1} e^{-76\theta}\, d\theta}_{1}$$

$$= \frac{(75)^{193.5}}{(76)^{193.5}} \frac{1}{x_{n+1}!}$$