Example: $X_i \sim N(\mu, \sigma^2)$, σ^2 is known

- Find the MLE of μ.
- $P(|\hat{\mu} - \mu| < 3 \mid \mu)$

x

a) we know that the M.L.E of μ, when σ^2 is known is

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

since $\hat{\mu}$ is a linear combination of Normal distributed r.v. the distr. of $\hat{\mu}$ is also normal.

$$E(\hat{\mu}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}\cdot n\mu = \mu$$

$$Var(\hat{\mu}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \left(\frac{1}{n}\right)^2 Var\left(\sum_{i=1}^{n} X_i\right) \overset{\text{indep.}}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

so $\hat{\mu} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.

- $P(|\hat{\mu} - \mu| < 3 \mid \mu) = P(-3 < \hat{\mu} - \mu < 3 \mid \mu)$

$$= P(\mu - 3 < \hat{\mu} < 3 + \mu \mid \mu)$$

$$= P(\hat{\mu} < 3 + \mu \mid \mu) - P(\hat{\mu} \le \mu - 3 \mid \mu).$$

$$\hat{\mu} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

b) • $P\left(\frac{|\hat{\mu} - \mu|}{\sqrt{\sigma^2/n}} < 15\right) = P\left(-15 < \frac{\hat{\mu} - \mu}{\sqrt{\sigma^2/n}} < 15\right)$

we have: $\hat{\mu} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\hat{\mu} - \mu \sim N(0, \sigma^2/n)$$

$$E(\hat{\mu} - \mu) = E(\hat{\mu}) - \mu = \mu - \mu = 0.$$

$$Var(\hat{\mu} - \mu) = Var(\hat{\mu}) + Var(\mu) = Var(\hat{\mu}) = \sigma^2/n$$

now we have that $\hat{\mu}-\mu \sim N(0, \sigma^2/n)$

$$\frac{\hat{\mu}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$$

$Var(aX) = a^2 Var(X)$

$$E\left(\frac{\hat{\mu}-\mu}{\sqrt{\sigma^2/n}}\right) = \frac{1}{\sqrt{\sigma^2/n}} E(\hat{\mu}-\mu) = \frac{1}{\sqrt{\sigma^2/n}} \cdot 0 = 0.$$

$$Var\left(\frac{\hat{\mu}-\mu}{\sqrt{\sigma^2/n}}\right) = \left(\frac{1}{\sqrt{\sigma^2/n}}\right)^2 Var(\hat{\mu}-\mu) = \frac{1}{\sigma^2/n} \cdot \sigma^2/n = 1.$$

$$\boxed{\frac{\hat{\mu}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1)}$$

so $P\left(-15 < \frac{\hat{\mu}-\mu}{\sqrt{\sigma^2/n}} < 15\right) = P(-15 < Z < 15)$, $Z \sim N(0,1)$

$= P(Z < 15) - P(Z \leq -15)$

$= \Phi(15) - \Phi(-15)$. ✓

$= 1$

χ^2 distribution:

$X \sim Gama(\alpha, \beta) \Rightarrow f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \; x>0$

if $\alpha = m/2$, $\beta = 1/2$ then

$$f_X(x) = \frac{(1/2)^{m/2}}{\Gamma(m/2)} x^{m/2-1} e^{-\frac{x}{2}}, \; x>0.$$

$$= \frac{1}{2^{m/2}\Gamma(m/2)} x^{\frac{m}{2}-1} e^{-\frac{x}{2}}, \; x>0$$

is the p.d.f. of the χ^2-distribution with m degrees of freedom.

$m = 2$: then

$$f_X(x) = \frac{1}{2\Gamma(1)} x^{1-1} e^{-\frac{x}{2}}, \; x>0$$

$$= \frac{1}{2} e^{-\frac{x}{2}}, \; x>0.$$

, is the p.d.f of an exponential distribution with parameter $1/2$.

Z is an integer. $\Gamma(z) = (z-1)!$

Properties of χ^2:

$X \sim Gama(a, b) \Rightarrow E(X) = \frac{a}{b}, \; Var(X) = \frac{a}{b^2}$

- χ^2 with m degrees of freedom is $Gama(\frac{m}{2}, \frac{1}{2})$

$Y \sim \chi^2(m)$

$$E(Y) = \frac{m/2}{1/2} = m.$$

$$Var(Y) = \frac{m/2}{(1/2)^2} = \frac{m}{2} \cdot \frac{4}{1} = 2m.$$

• Moment Generating Function: of the r.v. X is given by

$$E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x|\theta), & x \text{ is discrete.} \\ \int e^{tx} f(x|\theta)\,dx, & x \text{ is continuous.} \end{cases}$$

• for the χ^2 distribution:

$$E(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{(1/2)^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} x^{\frac{m}{2}-1} e^{-\frac{x}{2}}\,dx$$

$$= \frac{(1/2)^{m/2}}{\Gamma(m/2)} \int_0^\infty \underbrace{x^{\frac{m}{2}-1} e^{-x(\frac{1}{2}-t)}}_{\text{Gamma}(\frac{m}{2},\, \frac{1}{2}-t)}\,dx \quad \rightarrow \quad \frac{1}{2}-t>0$$

$$= \frac{(1/2)^{m/2}}{\cancel{\Gamma(m/2)}} \cdot \frac{\cancel{\Gamma(m/2)}}{(1/2-t)^{m/2}} \underbrace{\int_0^\infty \frac{(1/2-t)^{m/2}}{\Gamma(m/2)} x^{\frac{m}{2}-1} e^{-x(\frac{1}{2}-t)}\,dx}_{1}$$

$$= \left(\frac{1/2}{1/2-t}\right)^{m/2}$$

$$= \left(\frac{1}{1-2t}\right)^{m/2} \quad t < \frac{1}{2}$$

$$Z \sim \chi^2 \text{ with } \left(\frac{1}{1-2t}\right)^{\boxed{}/2}$$

$\boxed{}$ degrees of freedom