Example: $X_i \overset{iid}{\sim} \text{Bernoulli}(\theta)$, $i=1,\dots,n$.

a) p.f. of $T=\sum_{i=1}^{n} X_i$

$$f(x|\theta) = \theta^{x}(1-\theta)^{1-x} \quad , x\in\{0,1\}.$$

$$T=\sum_{i=1}^{n} X_i \sim \text{Binomial}(n,\theta)$$

[moment generating function $E(e^{Xs})$. X is r.v.]

the probability function of T is given by:

$$f(t|\theta) = \begin{cases} \binom{n}{t}\theta^{t}(1-\theta)^{n-t} & , t=0,1,\dots,n \quad , 0<\theta<1 \\ 0 & \text{otherwise.} \end{cases}$$

is still unknown

different values ~~of~~ of θ give us different $f(t|\theta)$.

b) $P(T<10|\theta) = P(\sum X_i <10|\theta)$

$$= P(\sum X_i \le 9|\theta)$$

$$= \sum_{t=0}^{9} f(t|\theta)$$

$$= \sum_{t=0}^{9} \binom{n}{t}\theta^{t}(1-\theta)^{n-t}$$

$$= \sum_{t=0}^{9} \binom{30}{t}\theta^{t}(1-\theta)^{30-t} \quad , 0<\theta<1$$

?

Def: sampling distr.

$$E_\theta(T) = \begin{cases} \sum_t t\, f(t|\theta) & \text{, if } t \text{ is discrete} \\ \int t\, f(t|\theta)\, dt & \text{, if } t \text{ is continuous} \end{cases}$$

$X_i \sim \text{Bernoulli}(\theta)$, $i=1,\dots,10$.

- Find the M.L.E. of θ

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

We know that $\sum_{i=1}^{n} X_i \sim \text{Bin}(n,\theta)$.

$$P(|\hat{\theta}-\theta| < 0.1 \mid \theta) = P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \theta\right| < 0.1 \mid \theta\right)$$

$$= P\left(-0.1 < \frac{1}{n}\sum_{i=1}^{n} X_i - \theta < 0.1 \mid \theta\right)$$

$$= P\left(\theta - 0.1 < \frac{1}{n}\sum_{i=1}^{n} X_i < \theta + 0.1 \mid \theta\right)$$

$$= P\left(n[\theta-0.1] < \sum_{i=1}^{n} X_i < n[\theta+0.1] \mid \theta\right), \quad n=10$$

where $\sum_{i=1}^{n} X_i \sim \text{Bin}(n,\theta)$.

$$= P\Big(10\theta - 1 < \underbrace{\sum X_i}_{\in\{0,1,2,\dots,10\}} < 10\theta + 1 \mid \theta\Big)$$

$$= P\left(\sum X_i = 10\theta \mid \theta\right)$$

b) <u>Bayes estimator:</u>

$X_i \mid \theta \sim \text{Bernoulli}(\theta)$

$\theta \sim \text{Beta}(10,10)$

$\Rightarrow \theta \mid \underset{\sim}{X} \sim \text{Beta}\left(10+\sum X_i,\ 10+n-\sum X_i\right)$

$X \sim \text{Beta}(a,b)$
$E(X) = \frac{a}{a+b}$

Under square error loss function

$$\delta^* = E(\theta \mid \underset{\sim}{X}) = \frac{10+\sum_{i=1}^{n} X_i}{20+n} = \frac{10+\sum_{i=1}^{10} X_i}{30}$$

$$P(|\delta^* - \theta| < 0.1 \mid \theta) = P(-0.1 < \delta^* - \theta < 0.1 \mid \theta)$$

$$= P\left(-0.1 < \frac{10+\sum X_i}{30} - \theta < 0.1 \,\middle|\, \theta\right)$$

$$= P\left(\theta - 0.1 < \frac{10+\sum X_i}{30} < \theta + 0.1 \,\middle|\, \theta\right)$$

$$= P\left(30[\theta-0.1] < 10+\sum X_i < 30[\theta+0.1] \,\middle|\, \theta\right)$$

$$= P\left(30[\theta-0.1]-10 < \sum_{i=1}^{10} X_i < 30[\theta+0.1]-10 \,\middle|\, \theta\right)$$

$$= P\left(30\theta - 13 < \underbrace{\sum_{i=1}^{10} X_i}_{\text{Bin}(n,\theta),\ n=10.} < 30\theta - 7 \,\middle|\, \theta\right)$$