Sampling Distributions:

(1). If $X_1, \dots, X_n$ form a random sample from a normal distr. with mean μ and variance σ^2, then any linear combination of $X_1 \dots X_n$, say $g(X_1 \dots X_n)$, has a normal distribution with mean $E(g(X_1 \dots X_n))$ and Variance is $Var(g(X_1 \dots X_n))$.

For instance $\overline{X_n} \sim N(\mu, \sigma^2/n)$. $\mu = E(\overline{X_n})$

$\frac{\sigma^2}{n} = Var(\overline{X_n})$.

(2) $\underline{\chi^2 \text{ distribution:}}$

if Y has a χ^2 distribution with m degrees of freedom then Y has a gamma distribution with parameter $a = \frac{m}{2}$ $b = \frac{1}{2}$. $E(Y) = m$, $Var(Y) = 2m$.

- if $Y_1, \dots, Y_K$ random variables with χ^2 distribution with m_i degrees of freedom, $i = 1, \dots, k$, and they are independent, then $\sum_{i=1}^{K} Y_i$ has a χ^2 distribution with $m_1 + \dots + m_k$ degrees of freedom.
- if X has a standard normal distribution then $X^2 \sim \chi^2(1)$

(3) <u>Joint distribution</u> of sample mean and sample variance:

if $X_1 \dots X_n$ are random variables ~~with~~ that are <u>normally distributed</u> with mean μ and variance σ^2 and they are independent, then.

$\overline{X_n}$ and $\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X_n})^2$ are independent random variables.

$$\left.\begin{array}{l} \overline{X_n} \sim N(\mu, \sigma^2/n) \\ \sum_{i=1}^{n}\left(\frac{X_i - \overline{X_n}}{\sigma}\right)^2 \sim \chi^2_{(n-1)} \end{array}\right] \text{ and they are independent.}$$

(9) t - distribution:

let Y be a random variable with a standard normal distribution and let Z be a random variable with χ^2 distribution with m degrees of freedom, and let Y and Z be independent.

Then
$$W = \frac{Y}{\sqrt{\frac{Z}{m}}}$$
has a t distribution with m degrees of freedom.

$$E(W) = 0 \quad , m > 1$$
$$Var(W) = \frac{m}{m-2} \quad , m > 2.$$

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if $X_1 \dots X_n$ that follow a normal distribution with mean μ and variance σ^2, then.

a) $\bar{X}_n \sim N(\mu, \sigma^2/n)$

b) $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$

c) $n\left(\frac{\bar{X}_n - \mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$

d) $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$

e) $\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2 \sim \chi^2_{(n-1)}$

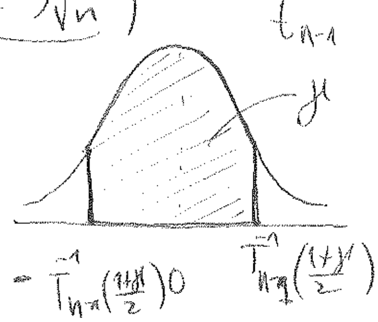
$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma'} \sim t_{(n-1)} \quad , \quad \sigma' = \sqrt{\frac{\sum(X_i - \bar{X}_n)^2}{n-1}}$$

Follows from b) and e)

confidence interval for the mean when $X_i \sim N(\mu, \sigma^2)$.

a coefficient γ confidence interval for μ, when σ^2 is unknown is given by

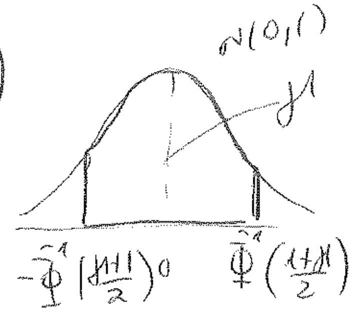
$$\Big(\underbrace{\bar{X}_n - T_{n-1}^{-1}\left(\tfrac{1+\gamma}{2}\right)\tfrac{\sigma'}{\sqrt{n}}}_{A} \;,\; \underbrace{\bar{X}_n + T_{n-1}^{-1}\left(\tfrac{1+\gamma}{2}\right)\tfrac{\sigma'}{\sqrt{n}}}_{B} \Big)$$



To find this interval we use the pivot $\dfrac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma'} \sim t_{n-1}$

a coefficient γ confidence interval for μ, when σ^2 is known is given by

$$\Big(\bar{X}_n - \Phi^{-1}\left(\tfrac{1+\gamma}{2}\right)\tfrac{\sigma}{\sqrt{n}} \;,\; \bar{X}_n + \Phi^{-1}\left(\tfrac{1+\gamma}{2}\right)\tfrac{\sigma}{\sqrt{n}} \Big)$$



To find this interval we use the pivot $\dfrac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$

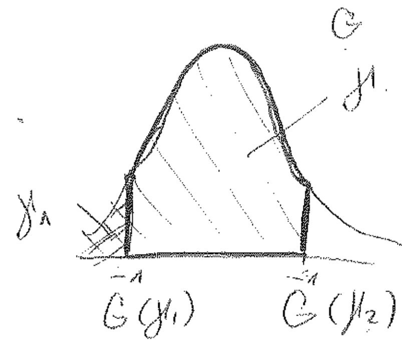
confidence interval for more general parameters

- pivotal: is a function of the random variables and a parameter of interest, say $g(\theta)$, that has a distribution that is the same for every θ. Lets denote the pivotal $V(\underset{\sim}{X}, g(\theta))$
- in order to use the pivotal for finding a confidence interval for $g(\theta)$ we need to find a function r such that

$$r\left(V(\underset{\sim}{X}, g(\theta)), \underset{\sim}{X} \right) = g(\theta).$$

— to find the interval for $g(\theta)$ we write.

$$P\left(G^{-1}(\gamma_1) < \underbrace{V(\underset{\sim}{X}, g(\theta))}_{\text{pivotal.}} < G^{-1}(\gamma_2) \right) = \gamma.$$



G is the C.D.F of $V(\underset{\sim}{X}, g(\theta))$.

$\gamma_2 - \gamma_1 = \gamma.$

and then find A and B such that

$$P(A < g(\theta) < B) = \gamma.$$

1) $X_1 \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$
$\quad Y \sim N(0, 4\sigma^2)$ $\Big>$ independent

a) find $g(X_1 \dots X_n, Y)$ such that has $t_{(n-1)}$

we know that if $Z \sim N(0,1)$ and $W \sim \chi^2_{(m)}$, then $\dfrac{Z}{\sqrt{\frac{W}{m}}} \sim t_m$.

$Y \sim N(0, 4\sigma^2) \Rightarrow \dfrac{Y}{\sqrt{4\sigma^2}} \sim N(0,1)$.

are independent

we know that
$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$, $\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2 \sim \chi^2_{(n-1)}$

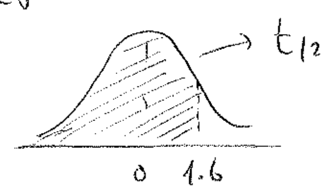
so
$$\frac{\frac{Y}{\sqrt{4\sigma^2}}}{\sqrt{\frac{\frac{\sum(X_i - \bar{X}_n)^2}{\sigma^2}}{n-1}}} = \frac{Y/2\sigma}{\frac{1}{\sigma}\sqrt{\frac{\sum(X_i - \bar{X}_n)^2}{(n-1)}}} = \frac{Y}{2\sqrt{\frac{\sum(X_i - \bar{X}_n)^2}{n-1}}} \sim t_{(n-1)}.$$

so $g(X_1 \dots X_n, Y) = \dfrac{Y}{2\sqrt{\frac{\sum(X_i - \bar{X}_n)^2}{n-1}}}$.

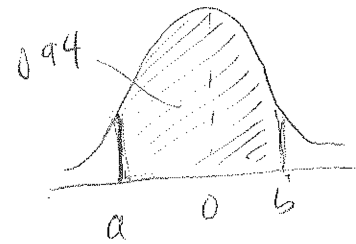
b) $P\left(\dfrac{Y}{2\sigma'} < 1.6\right)$, $\sigma' = \sqrt{\dfrac{\sum(X_i - \bar{X}_n)^2}{n-1}}$, $n = 13$.

from a) we have that $\dfrac{Y}{2\sigma'} \sim t_{12}$.

$P\left(\dfrac{Y}{2\sigma'} < 1.6\right) = 0.9322$



c) Find a, b such that $P(a < \frac{Y}{\underbrace{201}} < b) = 0.94$, $n = 13$



(a, b) :

we ~~else~~ can choose:

$\gamma_1 = 0.01$, $\gamma_2 = 0.95$

then $a = T_{n-1}^{-1}(0.01) = -2.68$

$b = T_{n-1}^{-1}(0.95) = 1.78$

we can also choose

$\gamma_1 = 0.02$ $\gamma_2 = 0.96$

then $a = T_{n-1}^{-1}(0.02) = -2.3$

$b = T_{n-1}^{-1}(0.96) = 1.91$

$\gamma_1 = 0.03$ $\gamma_2 = 0.97$

then $a = T_{n-1}^{-1}(0.03) = -2.03$

$b = T_{n-1}^{-1}(0.97) = 2.03$