Suppose that we observe $X_1 = x_1$, and $X_2 = x_2$ sequentially, so

$$\xi(\theta \mid x_1, x_2) \propto f(x_2 \mid \theta)\,\xi(\theta \mid x_1)$$

the normalizing constant is given by

$$\frac{1}{\int_\Omega f(x_2 \mid \theta)\,\xi(\theta \mid x_1)\,d\theta}$$

$$\int_\Omega f(x_2 \mid \theta)\,\xi(\theta \mid x_1)\,d\theta = \int_\Omega f(x_2 \mid \theta)\,\frac{f(x_1 \mid \theta)\,\xi(\theta)}{g(x_1)}\,d\theta$$

$$= \int_\Omega f(x_2, x_1 \mid \theta)\,\frac{\xi(\theta)}{g(x_1)}\,d\theta$$

$$= \int_\Omega \frac{h(x_2, x_1, \theta)}{g(x_1)}\,d\theta$$

$$= \frac{g(x_1, x_2)}{g(x_1)}$$

$$= h_1(x_2 \mid x_1)$$

so, if we observe $X_1 = x_1, \ldots, X_n = x_n$, then the normalizing constant would be

$$\frac{1}{\int_\Omega f(x_n \mid \theta)\,\xi(\theta \mid x_1 \ldots x_{n-1})\,d\theta} = \frac{1}{g(x_n \mid x_1 \ldots x_{n-1})}$$

if we now observe $X_{n+1} = x_{n+1}$, we have that

$$\frac{1}{\int f(x_{n+1} \mid \theta)\,\xi(\theta \mid x_1 \ldots x_n)\,d\theta} = \frac{1}{g(x_{n+1} \mid x_1 \ldots x_n)}$$, so

$$g(x_{n+1} \mid x_1 \ldots x_n) = \int f(x_{n+1} \mid \theta)\,\xi(\theta \mid x_1 \ldots x_n)\,d\theta.$$

Example: lifetime of a new component.

We have $X_i \mid \theta \overset{iid}{\sim} \exp(\theta)$, $i = 1, \dots, n$

$\theta \mid a, b \sim \text{Gama}(a, b)$

we know that $\quad X \sim \text{Gam}(a,b) \dashrightarrow \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$

$$\theta \mid x_1 \dots x_n \sim \text{Gama}\left(a+n, \; b + \sum_{i=1}^{n} x_i\right), \quad n = 5$$

$$P(X_6 > 1.5 \mid x_1 \dots x_5) = \int_{1.5}^{\infty} \underbrace{f(x_6 \mid x_1, \dots, x_5)}_{?} \, dx_6$$

$$f(x_6 \mid x_1 \dots x_5) = \int_{\Omega} f(x_6 \mid \theta) \, \xi(\theta \mid x_1 \dots x_5) \, d\theta$$

$$= \int_0^\infty \theta e^{-\theta x_6} \cdot \underbrace{\frac{(b + \sum x_i)^{a+n}}{\Gamma(a+n)}}_{\text{constant as a function of } \theta} \theta^{a+n-1} e^{-(b+\sum x_i)\theta} \, d\theta$$

$$= \frac{(b+\sum x_i)^{a+n}}{\Gamma(a+n)} \int_0^\infty \underbrace{\theta^{a+n} e^{-(b + \sum x_i + x_6)\theta}}_{\text{Gama}\left(a+n+1, \; b + \sum_{i=1}^{5} x_i + x_6\right)} \, d\theta$$

$\Gamma(n+1) = n\Gamma(n)$

$$= \frac{(b+\sum x_i)^{a+n}}{\Gamma(a+n)} \cdot \frac{\Gamma(a+n+1)}{\left(b + \sum_{i=1}^{5} x_i + x_6\right)^{a+n+1}}$$

$$= (a+n) \frac{(b + \sum x_i)^{a+n}}{(b + \sum x_i + x_6)^{a+n+1}}$$

you can check that

$$\int \frac{(a+n)(b+\sum x_i)^{a+n}}{\left(b + \sum_{i=1}^{5} x_i + x_6\right)^{a+n+1}} \, dx_6 = 1 \qquad \text{¡check it!}$$

$$\int y^{-10} \, dy$$

Finally

$(n=5)$

$$P(X_6 > 1.5 \mid x_1 \dots x_5) = \int_{1.5}^{\infty} \frac{(a+n)(b + \sum_1^5 x_i)^{a+n}}{(b + \sum x_i + x_6)^{a+n+1}} \, dx_6 = \dots = 0.1279.$$

b) $\mu_1 = \dfrac{\sigma^2 \mu_0 + n V_0^2 \overline{x_n}}{\sigma^2 + n V_0^2}$

$$= \underbrace{\frac{\sigma^2}{\sigma^2 + nV_0^2}} \mu_0 + \underbrace{\frac{nV_0^2}{\sigma^2 + nV_0^2}} \overline{x_n}.$$

- σ^2, V_0^2 are fixed, $n \to \infty$
- V_0^2, n fixed, $\sigma^2 \to \infty$
- σ^2, n fixed, $V_0^2 \to \infty$

c) $P(\theta > 1 \mid \underline{x})$, $\theta \mid \underline{x} \sim N(\mu_1, V_1^2)$

$$P(\theta > 1 \mid \underline{x}) = 1 - P(\theta \leq 1 \mid \underline{x})$$

$$= 1 - P\left(Z \leq \frac{1 - \mu_1}{\sqrt{V_1^2}}\right), \quad Z \sim N(0,1)$$

$$= 1 - \Phi\left(\frac{1-\mu_1}{\sqrt{V_1^2}}\right)$$

↳ cumulative distr. function of Z
(C.D.F.)