Example (Lifetime of components). (slide 9)

$X_1, \dots, X_5$

$X_i \mid \theta \overset{iid}{\sim} \exp(\theta)$

$$f(x_i \mid \theta) = \begin{cases} \theta e^{-\theta x_i}, & x_i > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0$$

$E(X_i \mid \theta) = E(X_i) = 1/\theta$

$Var(X_i \mid \theta) = Var(X_i) = 1/\theta^2$

$\theta \mid a, b \sim Gama(a, b)$, $a, b > 0$, a, b are known

$$\xi(\theta \mid a, b) = \xi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, \quad \theta > 0.$$

$E(\theta \mid a, b) = E(\theta) = \frac{a}{b}$

$Var(\theta \mid a, b) = Var(\theta) = \frac{a}{b^2}$.

a) find the posterior distribution of $\theta \mid x_1 \dots x_5$.

$$\xi(\theta \mid x_1 \dots x_n) = \frac{f_n(\underset{\sim}{x} \mid \theta)\, \xi(\theta)}{g_n(\underset{\sim}{x})}$$

- $f_n(\underset{\sim}{x} \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta)$, independence

$$= \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$= \theta^n e^{-\sum_{i=1}^{n} \theta x_i}$$

$$= \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

$a x_1 + a x_2$

$a(x_1 + x_2)$

- $g_n(\underset{\sim}{x}) = \int_0^\infty \theta^n e^{-\theta \sum_{i=1}^{n} x_i} \cdot \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \, d\theta$

$$= \frac{b^a}{\Gamma(a)} \int_0^\infty \underbrace{\theta^{a+n-1} e^{-(b + \sum_{i=1}^{n} x_i)\theta}}_{\text{similar to a Gamma distribution with parameters } a+n \text{ and } b + \sum_{i=1}^{n} x_i} d\theta$$

similar to a Gamma distribution with parameters $a+n$ and $b + \sum_{i=1}^{n} x_i$

$$\left(Y \sim \text{Gama}(a,b) \Rightarrow f(y|a,b) = \frac{b^a}{\Gamma(a)} y^{a-1} e^{-by} \right)$$

$$= \frac{b^a}{\Gamma(a)} \underbrace{\int_0^\infty \theta^{a+n-1} e^{-(b+\sum_{i=1}^n x_i)\theta} d\theta \cdot \frac{(b+\sum_{i=1}^n x_i)^{a+n}}{\Gamma(a+n)}}, \frac{\Gamma(a+n)}{(b+\sum_{i=1}^n x_i)^{a+n}}$$

$$= \frac{b^a}{\Gamma(a)} \cdot \frac{\Gamma(a+n)}{(b+\sum_{i=1}^n x_i)^{a+n}}$$

So

$$\xi(\theta|\underset{\sim}{x}) = \frac{\theta^n e^{-\theta\sum_{i=1}^n x_i} \cdot \cancel{\frac{b^a}{\Gamma(a)}} \theta^{a-1} e^{-b\theta}}{\cancel{\frac{b^a}{\Gamma(a)}} \frac{\Gamma(a+n)}{(b+\sum_{i=1}^n x_i)^{a+n}}}$$

$$= \frac{(b+\sum_{i=1}^n x_i)^{a+n}}{\Gamma(a+n)} \theta^{n+a-1} e^{-(b+\sum_{i=1}^n x_i)\theta}$$

So, we conclude that $\theta|\underset{\sim}{x} \sim \text{Gama}\left(a+n, b+\sum_{i=1}^n x_i\right)$

<u>Likelihood function:</u>

$$\xi(\theta|\underset{\sim}{x}) \propto f_n(\underset{\sim}{x}|\theta)\,\xi(\theta)$$

the normalizing constant can be found at any time doing:

If θ is discrete, then the normalizing constant is

$$\frac{1}{\sum_{\theta\in\Omega} f_n(\underset{\sim}{x}|\theta)\xi(\theta)}$$

if θ is continuous, then ...

$$\frac{1}{\int_\Omega f_n(\underset{\sim}{x}|\theta)\xi(\theta)\,d\theta}$$

Back to the lifetime example:

a) the likelihood function is $f_n(\underset{\sim}{x}|\theta)$ (as a function of θ!)

$$f_n(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \underbrace{\theta^n e^{-\theta\sum_{i=1}^{n} x_i}}_{\text{function of } \cancel{\text{theta}}\ \theta}, \quad n \text{ and } \sum_{i=1}^{n} x_i \text{ are observed.}$$

b) $\xi(\theta|\underset{\sim}{x}) \propto f_n(\underset{\sim}{x}|\theta)\,\xi(\theta)$

$$= \theta^n e^{-\theta\sum x_i} \underbrace{\frac{b^a}{\Gamma(a)}} \theta^{a-1} e^{-b\theta}$$

$$\propto \theta^{a+n-1} e^{-(b+\sum x_i)\theta}$$

So $\theta|\underset{\sim}{x} \sim G(a+n,\ b+\sum_{i=1}^{n} x_i)$

sequential observations

- suppose that we observe $X_1 = x_1$ that has p.d.f $f(x_1 \mid \theta)$ and we have $\xi(\theta)$.

$$\xi(\theta \mid x_1) \propto f(x_1 \mid \theta)\, \xi(\theta)$$

- suppose that now we observe $X_2 = x_2$ with is conditionally independent of X_1 given θ

$$\begin{aligned} \xi(\theta \mid x_1, x_2) &\propto f(x_1, x_2 \mid \theta)\, \xi(\theta) \\ &= f(x_2 \mid \theta)\, f(x_1 \mid \theta)\, \xi(\theta) \\ &\propto f(x_2 \mid \theta)\, \xi(\theta \mid x_1) \end{aligned}$$

$\vdots$

we have that

$$\xi(\theta \mid x_1 \ldots x_n) \propto f(x_n \mid \theta)\, \xi(\theta \mid x_1 \ldots x_{n-1})$$