Classical and Bayesian inference

AMS 132

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Independence of the Sample Mean and Sample Variance

- Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and unknown variance σ^2. Then, the M.L.E.s of μ and σ^2 are the sample mean $\overline{X}_n$ and the sample variance $(1/n)\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$.
- We will derive the joint distribution of these two estimators.

Independence of the Sample Mean and Sample Variance

Theorem

Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2. Then, the sample mean $\overline{X}_n$ and sample variance $(1/n)\sum_{i=1}^n (X_i - \overline{X}_n)^2$ are independent random variables, $\overline{X}_n$ has the normal distribution with mean μ and variance σ^2/n, and $\sum_{i=1}^n (X_i - \overline{X}_n)^2/\sigma^2$ has the χ^2 distribution with $n-1$ degrees of freedom.

Example (Lactic acid concentration in cheese)

One chemical whose concentration can affect taste is lactic acid. Cheese manufacturers who want to establish a loyal customer base would like the taste to be about the same each time a customer purchases the cheese. The variation in concentrations of chemicals like lactic acid can lead to variation in the taste of cheese.

Suppose that we model the concentration of lactic acid in several chunks of cheese as independent normal random variables with mean μ and variance σ^2.

We are interested in whether the sample variance underestimates the variance in the concentration of lactic acid in $k = 10$ chunks. For this, compute the probability that the sample variance underestimates the variance.

Estimation of the Mean and Standard Deviation

- Assume that $X_1, \ldots, X_n$ form a random sample from the normal distribution with unknown mean μ and unknown variance σ^2.
- Show that a sample size of $n = 21$ satisfy the following relationship

$$P\left(\mid \widehat{\mu} - \mu \mid \leq \frac{\sigma}{5} \ \text{ and } \ \mid \widehat{\sigma} - \sigma \mid \leq \frac{\sigma}{5} \right) \geq \frac{1}{2},$$

where $\widehat{\mu}$ and $\widehat{\sigma}$ are the M.L.E. of the mean and standard deviation, respectively.