University of California, Santa Cruz
Department of Applied Mathematics and Statistics
Baskin School of Engineering
Classical and Bayesian Inference - AMS 132

# Homework 5

**Instructions**: You have until Friday, March 16, to complete the assignment. It has to be returned during 10 first minutes of class (4:55 pm to 5:05 pm) or between 1:00 pm and 3:00 pm in office BE 357B.

Suppose that $X_1, \dots, X_n$ form a random sample from the Bernoulli distribution with probability of success $p$, $0 \leq p \leq 1$, where $X_i$ describes if tourist $i$ visiting Santa Cruz had a happy stay or not. Here we consider a happy stay as a success, so $p$ is the probability of a tourist having a happy stay in Santa Cruz.

1. Find the maximum likelihood estimator for $p$.

   The M.L.E. of $p$ is $\widehat{p} = \overline{X}_n$.

2. Get an expression for the mean and variance of the maximum likelihood estimator found in question 1.

   $E(\widehat{p}) = E(\overline{X}_n) = p$ and $Var(\widehat{p}) = Var(\overline{X}_n) = \frac{p(1-p)}{n}$.

3. Consider the following pivot $\frac{\sqrt{n}(\overline{X}_n - p)}{\sqrt{\overline{X}_n(1-\overline{X}_n)}}$, that has approximately a standard normal distribution. Find an approximate one-sided 90% confidence interval for the probability of a tourist having a happy stay in Santa Cruz. For this, find a lower bounded random variable $A$ such that $P(A \leq p) = 0.9$

   Since $\frac{\sqrt{n}(\overline{X}_n - p)}{\sqrt{\overline{X}_n(1-\overline{X}_n)}}$ has approximately a standard normal distribution, it follows that

$$P\left(\frac{\sqrt{n}(\overline{X}_n - p)}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} \leq \Phi^{-1}(0.9)\right) = 0.9,$$

   approximately. Now,

$$P\left(\frac{\sqrt{n}(\overline{X}_n - p)}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} \leq \Phi^{-1}(0.9)\right) = P\left(\overline{X}_n - p \leq \Phi^{-1}(0.9)\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}}\right),$$

$$= P\left(\overline{X}_n - \Phi^{-1}(0.9)\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}} \leq p\right).$$

   Therefore, an approximate random lower bound 90% confidence interval for $p$ is given by $(A, \infty)$, where $A = \overline{X}_n - \Phi^{-1}(0.9)\frac{\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}}$.

4. After asking 46 tourist, it was observed that 32 had a happy stay and 14 had not. Compute the (lower) one-sided 90% confidence interval for the probability of a tourist having a happy stay in Santa Cruz.

   From the data we have that $\overline{x}_n = 32/46$. Replacing this value in the confidence interval found in 3. we get that the approximate 90% lower confidence interval for $p$ is $(0.608, 1)$

5. The mayor, concerned about the attractiveness of Santa Cruz for visitors, decided that politics for advertising Santa Cruz will be considered if the probability that a visitors had a happy stay is less than one half. Based on the information provided in questions 1 to 4, what would the mayor do?

   Since with a 90% confidence the probability that a turist had a happy stayed in Santa Cruz is larger than 0.5, the mayor will decide not to advertise Santa Cruz as a tourist destination.

6. If only successes where observed, how would the interval computed in question 4. look like?

   If only successes are observed, we have that $\overline{x}_n = 1$ and the above interval is just 1, so based on that information we would say that all tourist have a happy stay in Santa cruz, and there is no uncertainty about that. This would be a very strong result!

7. Now, assume that the probability that a tourist in Santa Cruz had a happy stay follows a Beta distribution with parameters $a > 0$ and $b > 0$. Two advisers of the mayor have different beliefs about this probability. Both agree that the variance is similar, but they disagree in the mean. Adviser 1 beliefs that the prior for $p$ should have parameters $a = b = 10$, while adviser 2 thinks that $a = 10$ and $b = 2$. Describe the belief of both advisers in terms of the prior mean and prior variance.

   From the parameters from each prior we have that both advisers have the same belief about the variability of a tourist having a happy stay in SC, but, adviser 2 is more positive (here $E(p) = 0.833$) than adviser 1 (here $E(p) = 0.5$) in the tourist having a happy stay in SC.

8. Considering both prior distributions for $p$, find a value $c$ such that $P(c \leq p) = 0.9$, where $p \sim Beta(a, b)$. Considering only the prior belief of each adviser, would the mayor advertise Santa Cruz to get more tourists?
   For this, use the R command `qbeta(p, shape1=a, shape2=b)` that computes the $p-$th quantile of a Beta distribution with paramaters $a$ and $b$.

   Note that we need to find $c$ such that $P(c < p) = 0.9$ based on the prior distribution. This is the same as $P(p \leq c) = 0.1$. So we need to find the 0.1 quantile of the prior distribution for $p$ based on the belief of both advisers. Under adviser 1 we have that $c = 0.3579$ and under adviser 2 $c = 0.6897$. Therefore, based only on the prior belief of the advisers, the mayor will decide to advertise SC consider the prior belief of adviser 1, but will not advertise SC under the prior belief of adviser 2.

9. Considering general values $a$ and $b$ for the prior distribution of $p$, find the posterior distribution of the probability that a visitor had a happy stay.

   Since the beta prior is conjugate for Bernoulli sampling, we have that the posterior distribution of the probability that a visitor had a happy stay is a beta distribution with parameters $a + \sum_{i=1}^{n} x_i$ and $b + n - \sum_{i=1}^{n} x_i$.

10. Considering general values $a$ and $b$ for the prior distribution of $p$, and considering square error loss function, find Bayes estimator for $p$. Under what values of $a$ and $b$ would Bayes estimator and the maximum likelihood estimator be the same? What kind of prior distribution is this?

    Under squared error loss function Bayes estimator is the mean of teh posterior distribution, so $\delta^{\star} = \frac{a+\sum_{i=1}^{n} X_i}{a+b+n}$. The M.L.E. and Bayes estimators are the same when $a = b = 0$, which is a limiting case of the beta prior distribution. And this limiting case results in an improper prior distribution because here $\xi(p) \propto \frac{1}{p(1-p)}$ whose interval is infinity.

11. Get an expression for the mean and variance of Bayes estimator found in question 10.

    From question 10 we have that $p \mid \boldsymbol{x} \sim Beta(a + \sum_{i=1}^{n} x_i, b + n - \sum_{i=1}^{n} x_i)$, therefore under squared error loss function Bayes estimator, $\delta^{\star}$ is given by $\delta^{\star} = \frac{a+\sum_{i=1}^{n} X_i}{a+n+n}$. So,

$$E(\delta^{\star}) = E\left(\frac{a + \sum_{i=1}^{n} X_i}{a+b+n}\right) = \frac{a + \sum_{i=1}^{n} E(X_i)}{a+b+n} = \frac{a+np}{a+b+n}$$

$$Var(\delta^{\star}) = Var\left(\frac{a + \sum_{i=1}^{n} X_i}{a+b+n}\right) = \frac{1}{(a+b+n)^2}\sum_{i=1}^{n} Var(X_i) = \frac{np(1-p)}{(a+b+n)^2}.$$

12. Considering the prior distributions proposed by adviser 1 and the information in question 4, find a value $d$ such that the posterior probability of $p$ being larger than that value is 0.9. This is, find the 0.1 quantile of the posterior distribution. This gives you a 90% credible regions for p! Repeat using the prior of adviser 2.
    For this, use the `R` command `qbeta(p, shape1=a, shape2=b)`.

    Note that we need to find a value $d$ such that $P(d < p) = 0.9$, where $p \mid \boldsymbol{x} \sim Beta(a + \sum_{i=1}^{n} x_i, b + n - \sum_{i=1}^{n} x_i)$. Now, $P(d < p) = 1 - P(p \leq d)$. So we need to find a value $d$ such that $P(p \leq d) = 0.1$, this is, $d$ is the 0.1 quantile of the posterior distribution. Under advisor's 1 prior belief, $d$ is the 0.1 quantile of a $Beta(42, 24)$ distribution, so $d = 0.559$. Under advisor's 2 prior belief, $d$ is the 0.1 quantile of a $Beta(42, 16)$ distribution, so $d = 0.647$.

13. For the problem described in question 5, the results found in 12, and considering each prior, what would the mayor do?

Based on the Bayesian analysis performed under both priors, the mayor will decide not to advertise SC as a tourist destination.

14. If only successes where observed, how would the credible regions computed in question 12 look like?

    If only successes are observed the credible regions would be: under adviser's 1 belief, $(0.79, 1)$, and under adviser's 2 belief, $(0.933, 1)$.