University of California, Santa Cruz

Department of Applied Mathematics and Statistics

Baskin School of Engineering

Classical and Bayesian Inference - AMS 132

Homework 5

Instructions: You have until Friday, March 16, to complete the assignment. It has to be returned during 10 first minutes of class (4:55 pm to 5:05 pm) or between 1:00 pm and 3:00 pm in office BE 357B.

Suppose that $X_1, \ldots, X_n$ form a random sample from the Bernoulli distribution with probability of success p, $0 \leq p \leq 1$, where X_i describes if tourist i visiting Santa Cruz had a happy stay or not. Here we consider a happy stay as a success, so p is the probability of a tourist having a happy stay in Santa Cruz.

1. Find the maximum likelihood estimator for p.

2. Get an expression for the mean and variance of the maximum likelihood estimator found in question 1.

3. Consider the following pivot $\frac{\sqrt{n}(\overline{X}_n - p)}{\sqrt{\overline{X}_n(1 - \overline{X}_n)}}$, that has a standard normal distribution. Find a one-sided 90% confidence interval for the probability of a tourist having a happy stay in Santa Cruz. For this, if a lower bounded random variable A such that $P(A \leq p) = 0.9$

4. After asking 46 tourist, it was observed that 32 had a happy stay and 14 had not. Compute the (lower) one-sided 90% confidence interval for the probability of a tourist having a happy stay in Santa Cruz.

5. The mayor, concerned about the attractiveness of Santa Cruz for visitors, decided that politics for advertising Santa Cruz will be considered if the probability that a visitors had a happy stay is less than one half. Based on the information provided in questions 14, what would the mayor do?

6. If only successes where observed, how would the interval computed in question 4. look like?

7. Now, assume that the probability that a tourist in Santa Cruz had a happy stay follows a Beta distribution with parameters $a > 0$ and $b > 0$. Two advisers of the mayor have different beliefs about this probability. Both agree that the variance is similar, but they disagree in the mean. Adviser 1 beliefs that the prior for p should have parameters $a = b = 10$, while adviser 2 thinks that $a = 10$ and $b = 2$. Describe the belief of both advisers in terms of the prior mean and prior variance.

8. Considering both prior distributions for p, find a value c such that $P(c \leq p) = 0.9$, where $p \sim Beta(a, b)$. Considering only the prior belief of each adviser, would the mayor advertise Santa Cruz to get more tourists?
   For this, use the R command `qbeta(p, shape1=a, shape2=b)` that computes the $p-$th quantile of a Beta distribution with paramaters a and b.

9. Considering general values a and b for the prior distribution of p, find the posterior distribution of the probability that a visitor had a happy stay.

10. Considering general values a and b for the prior distribution of p, and considering square error loss function, find Bayes estimator for p. Under what values of a and b would Bayes estimator and the maximum likelihood estimator be the same? What kind of prior distribution is this?

11. Get an expression for the mean and variance of Bayes estimator found in question 10.

12. Considering the prior distributions proposed by adviser 1 and the information in question 4, find a value d such that the posterior probability of p being larger than that value is 0.9. This is, find the 0.1 quantile of the posterior distribution. This gives you a 90% credible regions for p! Repeat using the prior of adviser 2.
    For this, use the R command `qbeta(p, shape1=a, shape2=b)`.

13. For the problem described in question 5, the results found in 12, and considering each prior, what would the mayor do?

14. If only successes where observed, how would the credible regions computed in question 12 look like?