University of California, Santa Cruz
Department of Applied Mathematics and Statistics
Baskin School of Engineering
Classical and Bayesian Inference - AMS 132

Homework 4

Instructions: You have until Friday, March 2, to complete the assignment. It has to be returned during 10 first minutes of class (4:55 pm to 5:05 pm) or between 1:00 pm and 3:00 pm in office BE 357B.

1. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2, and let $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$. Determine the smallest values of n for which the following relations are satisfied:

   (a) $P\left(\frac{\widehat{\sigma}^2}{\sigma^2} \leq 1.9\right) \geq 0.95$.

   (b) $P\left(\mid \widehat{\sigma}^2 - \sigma^2 \mid \leq \frac{1}{2}\sigma^2\right) \geq 0.8$.

2. Suppose that the five random variables $X_1, \ldots, X_5$ are i.i.d. and that each has the standard normal distribution. Determine a constant c such that the random variable

$$\frac{c(X_1 + X_2)}{(X_3^2 + X_4^2 + X_5^2)^{1/2}},$$

   will have the t distribution.

3. Suppose that a random sample of eight observations is taken from the normal distribution with unknown mean μ and unknown variance σ^2, and that the observed values are $3.1, 3.5, 2.6, 3.4, 3.8, 3.0, 2.9$, and 2.2. Find a confidence interval for μ with each of the following three confidence coefficients: (a) 0.90, (b) 0.95, and (c) 0.99. What effect has the confidence coefficient in the size of the interval?

4. Suppose that $X_1, ..., X_n$ form a random sample from the normal distribution with unknown mean μ and unknown variance σ^2, and let the random variable L denote the length of the confidence interval for μ that can be constructed from the observed values in the sample. Find the value of $E(L^2)$ for the following values of the sample size n and the confidence coefficient γ:

   - for $\gamma = 0.95$: use $n = 5$, $n = 10$, and $n = 30$. For fixed γ, what effect has n in the size of the interval?
   - For $n = 8$: , use $\gamma = 0.90$,$\gamma = 0.95$, and $\gamma = 0.99$. For fixed n, what effect has the confidence coefficient γ?

   Note: $E(L^2)$ will be a function of σ^2. For solving the exercise: first find L and show that $L^2 = 4c^2\sigma'^2/n$, where $\sigma'^2 = \sum(X_i - \overline{X}_n)^2/(n-1)$. Second, note that $W = \sum(X_i - \overline{X}_n)^2/\sigma^2$ has a χ square distribution with $n-1$ degrees of freedom, whose mean is $n-1$.

5. Suppose that $X_1, ..., X_n$ form a random sample from the exponential distribution with unknown mean μ. Find a 90 percent confidence interval for μ. Use $\gamma_1 = 0.05$ and $\gamma_2 = 0.95$
Note: use the facts that: if $X \sim exp(\lambda)$, then $\sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$. If $X \sim Gamma(a, b)$, then $2Xb \sim \chi^2_{(2a)}$. Then, show that $\frac{2\sum_{i=1}^{n} X_i}{\mu}$ is a pivot quantity.

6. In the June 1986 issue of *Consumer Reports*, some data on the calorie content of beef hot dogs is given. Here are the numbers of calories in 20 different hot dog brands:
$186, 181, 176, 149, 184, 190, 158, 139, 175, 148, 152, 111, 141, 153, 190, 157, 131, 149, 135, 132.$
Assume that these numbers are the observed values from a random sample of twenty independent normal random variables with mean μ and variance σ^2, both unknown. Find a 90% confidence interval for the mean number of calories μ.

7. Consider the problem and data from exercise 6, but now assume that the variance is known and equal to 510.

   (a) Find a 90% confidence interval for the mean number of calories μ. You can use the result of exercise 1 in section 8.5 from the textbook (a confidence interval in this case was also solved in discussion section).

   (b) Now, assume that μ has a normal prior distribution with mean 0 and variance 10000. Find the posterior distribution of μ and compute the probability that the posterior distribution of μ lies between the confidence interval computed in a).

   (c) What kind of prior distribution was considered in b)?