University of California, Santa Cruz
Department of Applied Mathematics and Statistics
Baskin School of Engineering
Classical and Bayesian Inference - AMS 132

Homework 3

Instructions: You have until Wednesday, February 21, to complete the assignment. It has to be returned during 10 first minutes of class (4:55 pm to 5:05 pm) or between 1:00 pm and 3:00 pm in office BE 357B.

1. Suppose that a random sample is to be taken from the Bernoulli distribution with unknown parameter p. Suppose also that it is believed that the value of p is in the neighborhood of 0.2. How large a random sample must be taken in order that $P(| \overline{X}_n - p | \leq 0.1) \geq 0.75$ when $p = 0.2$?
   To do this, first find the distribution of $n\overline{X}_n$. Later, try values of n between 8 and 11. For computing the probabilities you can use: functions `dbinom(x, size=n, prob=p)` or `pbinom(x, size=n, prob=p)` from `R` that computes $P(X = x)$, or $P(X \leq x)$, respectively, when $X \sim Binomial(n, p)$; a calculator; or the table in the back of the book, page 854.

2. Considering that $X_1, \ldots, X_n$ form a random sample from the Bernoulli distribution with parameter p, use the central limit theorem in Sec. 6.3 to find approximately the size of a random sample that must be taken in order that$P(| \overline{X}_n - p | \leq 0.1) \geq 0.95$ when $p = 0.2$.
   Recall that the central limit theorem states that if $X_1, \ldots, X_n$ are independent and identically distributed random variable with mean μ and variance σ^2, then $\overline{X}_n \sim N(\mu, \sigma^2/n)$, for large enough n.

3. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2. Assuming that the sample size n is 16, determine the following probability
$$P\left(\frac{1}{2}\sigma^2 \leq \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 \leq 2\sigma^2\right).$$
   For computing the probabilities you can use: functions `dchisq(x, df=m)` or `pchisq(x, df=m)` from `R` that computes $f(x)$, or $P(X \leq x)$, respectively, when $X \sim \chi^2_{(m)}$; or the table in the back of the book, page 858.

4. Suppose that $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2, and let $\widehat{\sigma_0}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$. Determine the smallest values of n for which the following relations are satisfied:
   (a) $P\left(\frac{\widehat{\sigma_0}^2}{\sigma^2} \leq 1.9\right) \geq 0.95$.

(b) $P\left(\mid \widehat{\sigma_0}^2 - \sigma^2 \mid \leq \frac{1}{2}\sigma^2\right) \geq 0.8$.

5. Suppose that the random variables X_1, X_2, and X_3 are i.i.d., and that each has the standard normal distribution. Also, suppose that

$$Y_1 = 0.8X_1 + 0.6X_2,$$
$$Y_2 = \sqrt{2}(0.3X_1 - 0.4X_2 - 0.5X_3),$$
$$Y_3 = \sqrt{2}(0.3X_1 - 0.4X_2 + 0.5X_3).$$

   Find the distribution of Y_1, Y_2, and Y_3.

6. Suppose that a point (X, Y) is to be chosen at random in the xy-plane, where X and Y are independent random variables and each has the standard normal distribution. If a circle is drawn in the xy-plane with its center at the origin, what is the radius of the smallest circle that can be chosen in order for there to be probability 0.99 that the point (X, Y) will lie inside the circle?

7. Suppose that six random variables $X_1, \ldots, X_6$ form a random sample from the standard normal distribution, and let

$$Y = (X_1 + X_2 + X_3)^2 + (X_4 + X_5 + X_6)^2.$$

   Determine a value of c such that the random variable cY will have a χ^2 distribution.