University of California, Santa Cruz
Department of Applied Mathematics and Statistics
Baskin School of Engineering
Classical and Bayesian Inference - AMS 132

Homework 2

Instructions: You have until Friday, February 2, to complete the assignment. It has to be returned during 10 last minutes of class (4:55 pm to 5:05 pm) or between 1:00 pm and 3:00 pm in office BE 357B.

This homework includes a BONUS exercise. The score in that exercise can replace the score of ANY other exercise you choose, in this or another homework .

1. Suppose that the number of defects in a 1200-foot roll of magnetic recording tape has a Poisson distribution for which the value of the mean θ is unknown, and the prior distribution of θ is the gamma distribution with parameters α and β.

   (a) Show that the posterior mean is a weighted average of the prior mean and the sample mean, with weights denoted by γ_n, this means, the posterior mean can be written as $\gamma_n E(\theta) + (1 - \gamma_n)\overline{X}_n$, where $\overline{X}_n = \sum_{i=1}^{n} X_i/n$ and show that $\gamma_n \to 0$ as the sample size increases, this is, as $n \to \infty$.

   (b) Considering the square error loss function, find Bayes estimator for θ, and show that they form a consistent sequence of estimators of θ.

   (c) If the prior mean and prior variance are equal to 3, find the values of the parameters of the prior distribution.

   (d) When five rolls of this tape are selected at random and inspected, the numbers of defects found on the rolls are 2, 2, 6, 0, and 3. Find Bayes estimate for θ under square error loss.

   (e) Find Bayes estimate for the probability of a 1200-foot roll of magnetic recording tape having no defects.

2. Suppose that $X_1, \ldots, X_n$ form a random sample from an exponential distribution for which the value of the parameter θ is unknown.

   (a) Determine the maximum likelihood estimator and estimate of θ.

   (b) Determine the maximum likelihood estimator and estimate of the probability of observing a value equal or smaller than x_0, where $x_0 > 0$.

   (c) Determine the maximum likelihood estimator and estimate of the median of the distribution.

3. Suppose that $X_1, \ldots, X_n$ form a random sample from a gamma distribution with parameters a and θ, where a is known and θ is unknown. It is known that θ can only be 1, or 2, or 3, and $a = 3$. Six random variables are observed to be $0.5, 0.8, 1.2, 0.3, 1.4, 0.2$.

(a) Write the statistical model.

(b) Plot the likelihood function and plot the logarithm of the likelihood function.

(c) Find the maximum likelihood estimator and estimate of θ.

(d) Find the maximum likelihood estimator and estimate of $2\theta + 8$.

4. Suppose that $X_1, \dots, X_n$ form a random sample from a gamma distribution with parameters a and θ, where a is known and θ is unknown. Suppose that a discrete prior distribution for θ is assumed, where $\xi(1) = 0.3$, $\xi(2) = 0.5$, and $\xi(3) = 0.2$, and $a = 3$. Six random variables are observed to be $0.5, 0.8, 1.2, 0.3, 1.4, 0.2$.

   (a) Find the posterior distribution of θ.

   (b) Find Bayes estimate under absolute error loss function.

   (c) Find Bayes estimate under square error loss function.

   (d) Find Bayes estimate for $\Psi = 2\theta + 8$ under square error loss function.

5. (**Bonus exercise**): Find the distribution of a new observation given and observed sample of size n, $X_1 = x_1, \dots, X_n = x_n$, for the following models:

   - $X_1, \dots, X_n$ form a random sample from the Bernoulli distribution with parameter θ, and θ has a beta prior distribution with parameters $\alpha > 0$ and $\beta > 0$.
   - $X_1, \dots, X_n$ form a random sample from the Poisson distribution with parameter θ, and θ has a gamma prior distribution with parameters $\alpha > 0$ and $\beta > 0$.
   - $X_1, \dots, X_n$ form a random sample from the Normal distribution with unknown mean θ and known variance $\sigma^2 > 0$, and θ has a normal prior distribution with mean μ_0 and variance v_0^2.
   - $X_1, \dots, X_n$ form a random sample from the exponential distribution with parameter θ, and θ has a gamma prior distribution with parameters $\alpha > 0$ and $\beta > 0$.