University of California, Santa Cruz
Department of Applied Mathematics and Statistics
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Classical and Bayesian Inference - AMS 132

Homework 1: Solution

1. Suppose that the number of defects on a roll of magnetic recording tape has a Poisson distribution for which the mean θ is unknown. **[15 pts.]**

   (a) Write the statistical model. **[5 pts.]**
   Let X be the r.v. describing the number of defects on a roll of magnetic recording tape. The p.f. of the statistical model is $f(x \mid \theta) = e^{-\theta}\theta^x/x!$, $x = 0, 1, 2, \ldots$, and $\theta \in \Omega = \mathbb{R}^+$.

   (b) Assume that the mean number of defects on a roll of magnetic recording tape is either 1.0 or 1.5, and the prior p.f. of θ is $\xi(1.0) = 0.4$ and $\xi(1.5) = 0.6$. If a roll of tape selected at random is found to have three defects, what is the posterior p.f. of θ? **[5 pts.]**
   Here $\Omega = \{1, 1.5\}$, and $\xi(1.0) = 0.4$ and $\xi(1.5) = 0.6$.
   The marginal p.f. of X is given by

$$g(x) = \sum_{\theta\in\Omega} f(x \mid \theta)\xi(\theta) = \frac{e^{-1}(1)^x}{x!} \times 0.4 + \frac{e^{-1.5}(1.5)^x}{x!} \times 0.6.$$

   Then the posterior p.f. of θ given $X = 3$ is

$$\xi(1 \mid X = 3) = \left(\frac{e^{-1}}{3!} \times 0.4\right) \Big/ \left(\frac{e^{-1}(1)^3}{3!} \times 0.4 + \frac{e^{-1.5}(1.5)^3}{3!} \times 0.6\right) = 0.2457,$$

$$\xi(1.5 \mid X = 3) = 1 - \xi(1 \mid X = 3) = 0.7543.$$

   (c) Under the same assumptions of (b), find the conditional p.f. of X_2 given $X_1 = 3$. **[5 pts.]**
   The conditional p.f. of X_2 given $X_1 = 3$ is given by

$$f(x_2 \mid X_1 = 3) = \sum_{\theta\in\Omega} f(x_2 \mid \theta)\xi(\theta \mid X_1 = 3) = \frac{e^{-1}(1)^{x_2}}{x_2!} \times 0.2457 + \frac{e^{-1.5}(1.5)^{x_2}}{x_2!} \times 0.7543.$$

   Note that, $\sum_{x_2=0}^{\infty} f(x_2 \mid X_1 = 3) = 1$.

2. Suppose that a single observation X is to be taken from the uniform distribution on the interval $[\theta - 1/2, \theta + 1/2]$, where the value of θ is unknown, and the prior distribution of θ is the uniform distribution on the interval $[10, 20]$. If the observed value of X is 12, what is the posterior distribution of θ? **[15 pts.]**
   Since $\theta - 1/2 < x < \theta + 1/2$, it follows that $x - 1/2 < \theta < x + 1/2$. Additionally, $10 < \theta < 20$. Therefore,

$$\max\{x - 1/2, 10\} < \theta < \min\{x + 1/2, 20\}.$$

Since

$$\xi(\theta \mid x) \propto f(x \mid \theta)\xi(\theta) = 1 \times 1/10,$$

it follows that the posterior distribution of θ is constant on the interval $(\max\{x-1/2, 10\}, \min\{x+1/2, 20\})$. Finally, since $X = 12$ is observed, we have that the posterior distribution of θ given $X = 12$ is a uniform distribution on the interval $(11.5, 12.5)$.

3. Automobile engines emit a number of undesirable pollutants when they burn gasoline. One class of polutants consists of the oxides of nitrogen. Suppose that $X_1, \ldots, X_n$ describe the emissions of oxides of nitrogen, that can be described by a normal distribution with unknown mean θ and known variance σ^2. Also, assume that a normal prior p.d.f. for θ is considered with mean μ_0 and variance v_0^2. **[15 pts.]**

   (a) Show that the posterior distribution of θ is a normal distribution with mean and variance given by $\mu_1 = \frac{\sigma^2\mu_0 + nv_0^2\overline{x}_n}{\sigma^2 + nv_0^2}$, and $v_1^2 = \frac{\sigma^2 v_0^2}{\sigma^2 + nv_0^2}$. **[5 pts.]**

$$\begin{aligned}
\xi(\theta \mid \boldsymbol{x}) &\propto \prod_{i=1}^{n} \exp\left\{\frac{1}{2\sigma^2}(x_i - \theta)^2\right\} \exp\left\{\frac{1}{2v_0^2}(\theta - \mu_0)^2\right\}, \\
&\propto \exp\left\{-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{v_0^2}\right)\left[\theta^2 - 2\frac{(\sum_{i=1}^{n} x_i/\sigma^2 + \mu_0/v_0^2)}{\left(\frac{n}{\sigma^2} + \frac{1}{v_0^2}\right)}\theta\right]\right\}, \\
&\propto \exp\left\{-\frac{1}{2}\frac{1}{\frac{\sigma^2 v_0^2}{\sigma^2 + nv_0^2}}\left(\theta - \frac{\sigma^2\mu_0 + nv_0^2\overline{x}_n}{\sigma^2 + nv_0^2}\right)^2\right\}.
\end{aligned}$$

   Therefore, the posterior distribution of θ given $X_1 = x_1, \ldots, X_n = x_n$ is a normal distribution with mean μ_1 and v_1^2.

(b) Find the distribution of X_{n+1} given $X_1 = x_1, \ldots, X_n = x_n$. **[5 pts.]**

$$
\begin{aligned}
f(x_{n+1} \mid \boldsymbol{x}) &= \int_{-\infty}^{\infty} f(x_{n+1} \mid \theta)\xi(\theta \mid \boldsymbol{x})d\theta, \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_{n+1}-\theta)^2\right\} \frac{1}{\sqrt{2\pi v_1^2}} \exp\left\{-\frac{1}{2v_1^2}(\theta-\mu)^2\right\} d\theta, \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{2\pi v_1^2}} \exp\left\{-\frac{x_{n+1}^2}{2\sigma^2} - \frac{\mu_1^2}{2v_1^2}\right\} \times, \\
&\quad \int_{-\infty}^{\infty} \exp\left\{-\frac{(\theta^2 - 2x_{n+1}\theta)}{2\sigma^2} - \frac{(\theta^2 - 2\mu_2\theta)}{2v_1^2}\right\} d\theta, \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{2\pi v_1^2}} \exp\left\{-\frac{x_{n+1}^2}{2\sigma^2} - \frac{\mu_1^2}{2v_1^2}\right\} \times, \\
&\quad \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma^2+v_1^2}\right)\left[\theta^2 - 2\frac{(x_{n+1}/\sigma^2 + \mu_1/v_1^2)\,\theta}{\left(\frac{1}{\sigma^2+v_1^2}\right)}\right]\right\} d\theta, \\
&= \frac{1}{\sqrt{2\pi(\sigma^2+v_1^2)}} \exp\left\{-\frac{1}{2(\sigma^2+v_1^2)}(x_{n+1}-\mu_1)^2\right\}.
\end{aligned}
$$

Therefore, X_{n+1} given $X_1 = x_1, \ldots, X_n = x_n$ has a normal distribution with mean μ_1 and variance $\sigma^2 + v_1^2$.

(c) Assume that $\sigma = 0.5$, $\mu_0 = 2.0$, and $v_0 = 1.0$. In a random sample of $n = 46$ engines, the average of the emissions of oxides of nitrogen was 1.329. Compute $P(X_{47} < 1.5 \mid x_1, \ldots, x_n)$. **[5 pts.]**

With the given information, $\mu_1 = 1.332627$ and $v_1^2 = 0.005405405$, hence

$$
P(X_{47} < 1.5 \mid x_1, \ldots, x_n) = \Phi(Z < (1.5 - 1.332627)/\sqrt{0.5^2 + 0.005405405}) = 0.6297475,
$$

where $Z \sim N(0,1)$ and Φ denotes the cumulative distribution function.

4. Suppose that $X_1, \ldots, X_n$ form a random sample from a distribution for which the p.d.f. $f(x \mid \theta)$ is $f(x \mid \theta) = \theta e^{-\theta x}$, for $x > 0$, and $f(x \mid \theta) = 0$, otherwise. Suppose that the value of the parameter θ is unknown ($\theta > 0$), and the prior distribution of θ is the gamma distribution with parameters α and β ($\alpha > 0$ and $\beta > 0$). **[15 pts.]**

If $Y \mid a, b \sim Gamma(a, b)$, then $f(y \mid a, b) = \frac{b^a}{\Gamma(a)} y^{a-1} e^{-by}$, $E(Y \mid a, b) = a/b$ and $Var(Y \mid a, b) = a/b^2$.

(a) Write the statistical model. **[5 pts.]**

Let X be the r.v. of interest. The p.d.f. of the statistical model is $f(x \mid \theta) = \theta e^{-\theta x}$, $x > 0$, and $\theta \in \Omega = \mathbb{R}^+$.

(b) Determine the mean and the variance of θ after observing $X_1 = x_1, \ldots, X_n = x_n$. **[5 pts.]**

First we find the posterior distribution of θ given the observations and then its mean and variance.

$$
\xi(\theta \mid \boldsymbol{x}) \propto \theta^n e^{-\theta \sum_{i=1}^n x_i} \theta^{\alpha-1} e^{-\beta\theta} \propto \theta^{\alpha+n-1} e^{-(\beta + \sum_{i=1}^n x_i)\theta}.
$$

Therefore $\theta \mid \boldsymbol{x} \sim Gamma(\alpha + n, \beta + \sum_{i=1}^n x_i)$. So, $E(\theta \mid \boldsymbol{x}) = (\alpha + n)/(\beta + \sum_{i=1}^n x_i)$ and $Var(\theta \mid \boldsymbol{x}) = (\alpha + n)/(\beta + \sum_{i=1}^n x_i)^2$.

(c) If $\alpha = \beta = 0$, is the prior p.d.f. of θ improper? Is the posterior p.d.f. of θ improper? **[5 pts.]**
If $\alpha = \beta = 0$, then $\xi(\theta) \propto \theta^{-1}$ and $\int_0^\infty \theta^{-1} d\theta = ln(\theta)|_{\theta=0}^\infty = \infty$, so the prior p.d.f. of θ given the observations is improper. The posterior distribution of θ given the observations is not improper as long as $\sum_{i=1}^n x_i > 0$ and $n \geq 1$.

5. Show that the family of beta distributions with parameters $a, b > 0$, is a conjugate family of prior distributions for samples from a negative binomial distribution with a known value of the parameter r and an unknown value of the parameter p $(0 < p < 1)$. **[15 pts.]**
If $X \mid p \sim NegativeBinomial(r, p)$, then $P(X = x \mid p) \propto p^r(1-p)^x$.
The posterior distribution of p given the observations is

$$\xi(p \mid \boldsymbol{x}) \propto \prod_{i=1}^n \{p^r(1-p)^{x_i}\} p^{a-1}(1-p)^{b-1} = p^{a+nr-1}(1-p)^{b+\sum_{i=1}^n x_i - 1}.$$

So, we recognize the kernel of a beta distribution with parameters $a + nr$ and $b + \sum_{i=1}^n x_i$, and so, the family of beta distributions is a conjugate family of prior distributions for samples from a negative binomial distribution

6. Suppose that the time in minutes required to serve a customer at a certain facility has an exponential distribution for which the value of the parameter θ is unknown and that the prior distribution of θ is a gamma distribution. **[15 pts.]**
If $Y \mid a, b \sim Gamma(a, b)$, then $f(y \mid a, b) = \frac{b^a}{\Gamma(a)} y^{a-1} e^{-by}$, $E(Y \mid a, b) = a/b$ and $Var(Y \mid a, b) = a/b^2$.

(a) The average time required to serve a random sample of 20 customers is observed to be 3.8 minutes. If the prior distribution of θ has mean equal to 0.2 and the standard deviation equal to 1, what is the posterior distribution of θ?**[5 pts.]**
Let X be a r.v. describing the time in minutes required to serve a customer at a certain facility. Then $X \mid \theta \sim exp(\theta)$, and we are assuming that $\theta \mid a, b \sim Gamma(a, b)$. We know that the gamma prior is conjugate for the exponential distribution, therefore $\theta \mid \boldsymbol{x} \sim Gamma(a + n, b + \sum_{i=1}^n x_i)$
Since $a/b = 0.2$ and $a/b^2 = 1^2$, it follows that $a = 0.04$ and $b = 0.2$. Additionally, $n = 20$ and $\sum_{i=1}^n x_i = 20 \times 3.8 = 76$, so we conclude that,

$$\theta \mid \boldsymbol{x} \sim Gamma(20.04, 76.2).$$

(b) For a distribution with mean $\mu \neq 0$ and standard deviation $\sigma > 0$, the *coefficient of variation* of the distribution is defined as $\sigma / \mid \mu \mid$. Suppose that the coefficient of variation of the prior gamma distribution of θ is 2. What is the smallest number of customers that must be observed in order to reduce the coefficient of variation of the posterior distribution to 0.1 or less? **[5 pts.]**

Since the gamma prior is conjugate for the exponential distribution, we have that $\theta \mid \boldsymbol{x} \sim Gamma(a+n, b+\sum_{i=1}^{n} x_i)$.

Since the coefficient of variation of the prior distribution is 2, it follows that $\frac{\sqrt{a/b^2}}{|a/b|} = \frac{1}{\sqrt{a}} = 2$, therefore $a = 0.25$. Now, for the posterior distribution of θ given the observations, we have that the coefficient of variation is given by $\frac{1}{\sqrt{a+n}}$. Therefore, $1/\sqrt{0.25+n} \leq 0.1$ implies that at least 100 customers must be observed.

(c) Suppose now that the coefficient of variation of the prior gamma distribution of θ is 0.2. What is the smallest number of customers that must be observed in order to reduce the coefficient of variation of the posterior distribution to 0.1? **[5 pts.]**

Since the coefficient of variation of the prior distribution is 0.2, it follows that $a = 0.0025$. Now, $1/\sqrt{0.0025+n} \leq 0.1$ implies that at least 75 customers must be observed.