Fisher information:

a) $X_i \sim \text{Bernoulli}(p) \qquad 0<p<1$

$$I(p) = \frac{1}{p(1-p)}$$

b) $X_i \sim N(\mu, \sigma^2)$, σ^2 is known.

$-\infty < \mu < \infty$

$$I(\mu) = -E\left[\frac{d^2}{d\mu^2} \log f(x|\mu)\right]$$

$$f(x|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-1}{2\sigma^2}(x-\mu)^2\right\}$$

$$\lambda(x|\mu) = \log f(x|\mu) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\frac{d \log f(x|\mu)}{d\mu} = \frac{\cancel{2}(x-\mu)}{\cancel{2}\sigma^2} = \frac{x-\mu}{\sigma^2}$$

$$\frac{d^2 \log f(x|\mu)}{d\mu^2} = \frac{-1}{\sigma^2}$$

$$I(\mu) = -E\left[\frac{d^2}{d\mu^2} \log f(x|\mu)\right] = -E\left[\frac{-1}{\sigma^2}\right] = \frac{1}{\sigma^2}$$

Example: customer arrivals.

a) n is fixed and X is a r.v. that describes the time until the n customers arrive.

$$X \sim \text{Gama}(n, \lambda)$$

b) t is fixed and X is a r.v. that describes the number of customers that arrive to the store ~~between~~ in an interval of time of length t.

$$X \sim \text{Poisson}(\lambda t)$$

Which sampling mechanism is better?

under scenario a) the Fisher information for λ is given by $I(\lambda) = \frac{n}{\lambda^2}$; exercise: show this!

Under scenario b) the Fisher information for λ is given by $I(\lambda) = \frac{t}{\lambda}$.

- if $n = t = 10$, b) contains more information about λ than a) for $\lambda > 1$.
- when do ~~both~~ a) and b) contain the same information about λ?

$$\frac{t}{\lambda} = \frac{n}{\lambda^2} \Rightarrow \lambda^2 t = n\lambda \Rightarrow \lambda t = n$$

under scenario a) we have that $X \sim \text{Gam}(n, \lambda)$ so $E(X) = \frac{n}{\lambda} \Rightarrow \lambda E(X) = n$

Under scenario $b)$ we have that $X \sim \text{Poisson}(\lambda t)$.
so $E(X) = \lambda t$

So if the expected time that we wait and the expected number of customers ~~is~~ are constant then the Fisher information for both sampling mechanisms is the same.

Efficient estimators:

a) $X_i \sim \text{Bernoulli}(p)$. Show that $\bar{X}_n$ is an efficient estimator for p. Thus, i.e., $Var(\bar{X}_n) = \frac{1}{nI(p)}$ and we need to check that $\bar{X}_n$ is unbiased.

- $E(\bar{X}_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = p.$
- $Var(\bar{X}_n) = Var\left(\frac{1}{n}\sum X_i\right) = \frac{p(1-p)}{n}$
- ~~we~~ $I(p) = \frac{1}{p(1-p)}$. $I_n(p) = nI(p) = \frac{n}{p(1-p)}$

since $Var(\bar{X}_n) = \frac{p(1-p)}{n} = \frac{1}{nI(p)}$ we have that $\bar{X}_n$ is an efficient estimator for p.

b) $X_i \sim N(\mu, \sigma^2)$ and σ^2 known.

$\bar{X}_n$ is ~~the~~ the estimator for μ.

$E(\bar{X}_n) = \mu$ so $\bar{X}_n$ is unbiased.

$Var(\bar{X}_n) = \frac{\sigma^2}{n}$

and $I_n(\mu) = nI(\mu) = \frac{n}{\sigma^2}$

since $Var(\bar{X}_n) = \frac{\sigma^2}{n} = \frac{1}{nI(\mu)}$, we have that $\bar{X}_n$

is an efficient estimator for μ.

<u>Asymptotic distribution of the M.L.E:</u>

If $\hat{\theta}_n$ is the M.L.E. of θ. then

$$\hat{\theta}_n \sim N\left(\theta, \frac{1}{nI(\theta)}\right), \quad n \text{ large}$$

$$\frac{\hat{\theta}_n - \theta}{\sqrt{\frac{1}{nI(\theta)}}} = \sqrt{nI(\theta)}\,(\hat{\theta}_n - \theta) \sim N(0,1), \quad n \text{ large.}$$