$$\boxed{\begin{array}{l}2\lambda\sum X_i \\ \sim \chi^2_{(2n)}\end{array}}$$

c) ~~...~~ 90% conf. interval for $\frac{1}{\lambda}$ is given by

$$(A, B) = \left(\frac{2\sum X_i}{G^{-1}(0.95)} , \frac{2\sum X_i}{G^{-1}(0.05)} \right) \text{, where}$$

$G^{-1}(0.05)$ and $G^{-1}(0.95)$ are the 0.05 and 0.95 quantiles of a χ^2 distr. with $2 \cdot n$ degrees of freedom.

d) $\lambda \sim \text{Gamma}(\alpha, \beta)$, $\alpha = 1$ $\beta = 0.25$

if $c = 0.08$ and $d = 4.87$ then

$$P\left(c < \frac{1}{\lambda} < d\right) = 0.9$$

e) first we find the posterior distn of $\underline{\lambda}$.

$X_i \sim \exp(\lambda)$

$\lambda \sim \text{Gamma}(\alpha, \beta)$.

$$\xi(\lambda \mid \underline{x}) \propto f(\underline{x} \mid \lambda)\,\xi(\lambda)$$

$$\propto \underline{\lambda^n e^{-\lambda \sum x_i}} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \underline{\lambda^{\alpha-1}}\, \underline{e^{-\beta\lambda}}.$$

$$\propto \lambda^{\alpha+n-1} \cdot e^{-[\beta + \sum x_i]\lambda}.$$

So, $\lambda \mid \underline{x} \sim \text{Gama}(\alpha+n, \underbrace{\beta + \sum x_i})$

therefor $2 \cdot \lambda \cdot [\beta + \sum x_i] \big| \underline{x} \sim \chi^2_{(2(\alpha+n))}$

$$P\left(G^{-1}(0.05) < 2\lambda[\beta + \sum x_i] < G^{-1}(0.95) \,\Big|\, \underline{x} \right) = 0.9$$

$$P\left(\frac{G^{-1}(0.05)}{2[\beta+\sum x_i]} < \lambda < \frac{G^{-1}(0.95)}{2[\beta + \sum x_i]} \,\Big|\, \underline{x} \right) = 0.9.$$

$$P\left(\underbrace{\frac{2[\beta+\sum x_i]}{G^{-1}(0.95)}} < \frac{1}{\lambda} < \underbrace{\frac{2[\beta+\sum x_i]}{G^{-1}(0.05)}} \,\Big|\, \underline{x} \right) = 0.9$$

So 90% credible region for $\frac{1}{\lambda}$, the mean monthly income, is given by

$$\left(\frac{2[\beta + \sum x_i]}{G^{-1}(0.95)} , \frac{2[\beta + \sum x_i]}{G^{-1}(0.05)} \right) = \left(\frac{2 \cdot [0.25 + 144.3]}{67.5} , \frac{2[0.25 + 144.3]}{34.76} \right)$$

$$= (4.24, 7.93)$$

$G^{-1}(0.95)$ and $G^{-1}(0.05)$ are the 0.95 and 0.05 quantiles of a χ^2 distribution with $2(\alpha + n) = 2(1 + 25)$ degrees of freedom.

<u>Unbiased estimators</u>

<u>Example</u>: $X_i \sim N(\theta, \sigma^2)$, θ, σ^2 unknown.

a) show $\sigma'^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ is unbiased for σ^2

for this we need to show that

$$E_{\sigma^2}[\sigma'^2] = \sigma^2$$

remember that $\sum_{i=1}^{n}\left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2 \sim \chi^2(n-1)$.

so $E\left[\sum_{i=1}^{n}\left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2\right] = n-1$.

now

$$E_{\sigma^2}[\sigma'^2] = E_{\sigma^2}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right] = E_{\sigma^2}\left[\underbrace{\frac{\sigma^2}{n-1}}_{\text{constant}}\sum_{i=1}^{n}\left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2\right]$$

$$= \frac{\sigma^2}{n-1}(n-1) = \sigma^2.$$

So, σ'^2 is an unbiased estimator for σ^2.

b) show that $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ is a <u>biased</u> estimator for σ^2. We will show that $E_{\sigma^2}[\hat{\sigma}^2] \neq \sigma^2$

$$E_{\sigma^2}[\hat{\sigma}^2] = E_{\sigma^2}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right] = E_{\sigma^2}\left[\frac{\sigma^2}{n}\sum_{i=1}^{n}\left(\frac{X_i - \bar{X}_n}{\sigma}\right)^2\right]$$

$$= \frac{\sigma^2}{n}(n-1) = \frac{(n-1)\sigma^2}{n}$$

So $\hat{\sigma}^2$ is an biased estimator for σ^2 and the bias is given by.

$$E_\sigma[\hat{\sigma}^2] - \sigma^2 = \frac{(n-1)\sigma^2}{n} - \sigma^2.$$