These questions were covered in details during the discussion sections. These are only partial solutions. Homeworks and exams should contain complete solutions to obtain full credit.

1. Let $\xi(\theta)$ be a p.d.f. that is defined as follows for constants $\alpha > 0$ and $\beta > 0$:

$$\xi(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta} & \text{for } \theta > 0, \\ 0 & \text{for } \theta \le 0. \end{cases}$$

A distribution with the p.d.f. is called an *inverse gamma distribution* denoted by $IG(\alpha, \beta)$, where α is the shape parameter and β is the scale parameter.

   (a) Let $\eta = 1/\theta$. Show that the p.d.f. of η is gamma distribution with the shape parameter α and rate parameter β, $Ga(\alpha, \beta)$.

   (b) Consider the family of probability distributions that can be represented by a p.d.f. $\xi(\theta)$ having the given form for all possible pairs of constants $\alpha > 0$ and $\beta > 0$. Show that this family is a conjugate family of prior distributions for samples from a normal distribution with a known value of the mean μ and an unknown value of the variance θ.

2. Suppose that $X_1, ..., X_n$ form a random sample from the exponential distribution with rate parameter θ. Let the prior distribution of θ be improper with "p.d.f." $1/\theta$ for $\theta > 0$. Find the posterior distribution of θ and show that the posterior mean of θ is $1/\overline{x}_n$.

3. Consider a distribution for which the p.d.f. or the p.f. is $f(x|\theta)$, where θ belongs to some parameter space Ω. It is said that the family of distributions obtained by letting θ vary over all values in Ω is an *exponential family*, if $f(x|\theta)$ can be written as follows for $\theta \in \Omega$ and all values of x:

$$f(x|\theta) = a(\theta)b(x)\exp[c(\theta)d(x)].$$

Here $a(\theta)$ and $c(\theta)$ are arbitrary functions of θ, and $b(x)$ and $d(x)$ are arbitrary functions of x. Let

$$H = \left\{(\alpha, \beta) : \int_\Omega a(\theta)^\alpha \exp[c(\theta)\beta]\, d\theta < \infty\right\}.$$

For each $(\alpha, \beta) \in H$, let

$$\xi_{\alpha,\beta}(\theta) = \frac{a(\theta)^\alpha \exp[c(\theta)\beta]}{\int_\Omega a(\eta)^\alpha \exp[c(\eta)\beta]\, d\eta},$$

and let Ψ be the set of all probability distributions that have p.d.f.'s of the form $\xi_{\alpha,\beta}(\theta)$ for some $(\alpha, \beta) \in H$.

   (a) Show that inverse gamma distribution $IG(\alpha, \theta)$ for which α is known and θ is unknown is a member of exponential family.

   (b) Suppose that we observe a random sample of size n from the distribution with p.d.f. $f(x|\theta)$. If the prior p.d.f. of θ is $\xi_{\alpha_0,\beta_0}(\theta)$, show that the posterior hyperparameters are

$$\alpha_1 = \alpha_0 + n,\ \beta_1 = \beta_0 + \sum_{i=1}^n d(x_i).$$

   That is, Ψ is a conjugate family of prior distributions for samples from $f(\boldsymbol{x}|\theta)$.