These questions were covered in details during the discussion sections. These are only partial solutions. Homeworks and exams should contain complete solutions to obtain full credit.

1. Suppose that the heights of men in a certain population follow a normal distribution with mean μ and variance 9. Assume we do not know the value of μ, and we wish to learn about it by sampling from the population. Suppose we decide to sample $n = 36$ men and let $\bar{X}_n$ stand for the average of their heights. Then the interval
$$(\bar{X}_n - 0.98, \bar{X}_n + 0.98)$$
has the property that it will contain the value of μ with propability 0.95.
   (a) Identify the components of the statistical model as defined in Definition 7.1.1.
   (b) Identify any statistical inference mentioned.
2. Suppose that the prior distribution of some parameter θ is a beta distribution for which the mean is 1/3 and the variance is 1/45. Determine the prior p.d.f. of θ
3. Suppose that the proportion θ of defective items in a large manufactured lot is unknown, and the prior distribution of θ is the uniform distribution on the interval [0,1]. When eight items are selected at random from the lot, it is found that exactly three of them are defective. Determine the posterior distribution of θ.

SOLUTIONS:

1. (a) The random variables of interest are the observable heights $X1, ..., Xn$, the hypothetically observable mean (parameter) μ, and the sample mean $\bar{X}_n$. The X_i's are modeled as normal random variables with mean μ and variance 9 given μ.
   (b) The statement that the interval $(\bar{X}_n - 0.98, \bar{X}_n + 0.98)$ has probability 0.95 of containing μ is an inference.
2. Let α and β denote the parameters of the beta distribution, then we know that:
$$\frac{\alpha}{\alpha+\beta} = \frac{1}{3} \quad \text{and} \quad \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{45}$$
Furthermore, we know that
$$\frac{\beta}{\alpha+\beta} = \frac{2}{3}.$$
Therefore the second equation reduces to:
$$\frac{2}{9(\alpha+\beta+1)} = \frac{1}{45}$$
Finally, we find that $\alpha+\beta+1 = 10$. Using $\alpha+\beta = 9$ in the first equation, we obtain $\alpha = 3$ and $\beta = 6$. Hence, the prior p.d.f. of θ is a Beta(3,6). That is, for $0 < \theta < 1$:
$$\xi(\theta) = \frac{\Gamma(9)}{\Gamma(3)\Gamma(6)}\theta^2(1-\theta)^5$$

3. To find the posterior distribution of θ, we need the prior distribution of θ, the distribution function of the data given θ, and finally the marginal distribution of the data. Starting with the prior, we have:

$$\xi(\theta) = 1, \;\; 0 < \theta < 1.$$

The distribution function is:

$$f(y|\theta) = \binom{8}{y} \theta^y (1-\theta)^{8-y}.$$

To find the marginal distribution, we need to integrate the distribution function with respect to θ. Recognizing the beta distribution kernel, we can solve this integral as follows:

$$\begin{aligned} g(y) &= \int_0^1 \binom{8}{y} \theta^y (1-\theta)^{8-y} d\theta \\ &= \binom{8}{y} \frac{\Gamma(y+1)\Gamma(9-y)}{\Gamma(10)} \int_0^1 \frac{\Gamma(10)}{\Gamma(y+1)\Gamma(9-y)} \theta^y (1-\theta)^{8-y} d\theta \\ &= \binom{8}{y} \frac{\Gamma(y+1)\Gamma(9-y)}{\Gamma(10)}. \end{aligned}$$

Putting it all together, we obtain the posterior distribution for θ:

$$\begin{aligned} \xi(\theta|y) &= \frac{f(y|\theta)\xi(\theta)}{g(y)} \\ &= \frac{\binom{8}{y}\theta^y(1-\theta)^{8-y}}{\binom{8}{y}\frac{\Gamma(y+1)\Gamma(9-y)}{\Gamma(10)}} \\ &= \frac{\Gamma(10)}{\Gamma(y+1)\Gamma(9-y)} \theta^y (1-\theta)^{8-y}. \end{aligned}$$

Including $y = 3$, we obtain the final answer:

$$\xi(\theta|y) = \frac{\Gamma(10)}{\Gamma(4)\Gamma(6)} \theta^3 (1-\theta)^5.$$

The posterior distribution is a Beta(4,6).

The End.