Classical and Bayesian inference

AMS 132

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Sampling from a Bernoulli Distribution

Theorem (Beta-Bernoulli model)

Suppose that $X_1, \ldots, X_n$ *form a random sample from the Bernoulli distribution with parameter* θ*, which is unknown* $(0 < \theta < 1)$*. Suppose also that the prior distribution of* θ *is the beta distribution with parameters* $\alpha > 0$ *and* $\beta > 0$*. Then the posterior distribution of* θ *given that* $X_i = x_i, (i = 1, ..., n)$*, is the beta distribution with parameters* $\alpha + \sum_{i=1}^n x_i$ *and* $\beta + n - \sum_{i=1}^n x_i$*.*

Example (Beta-Bernoulli model)

Consider a large shipment of iPhone has arrive, and the proportion of defective iPhone, θ, is unknown. A *Beta*(α, β) prior is assumed and *n* randomly selected items are inspected.

a) Find the posterior distribution of θ.
b) How is the posterior distribution updated?

Sampling from a Bernoulli Distribution

Definition (Conjugate Family and Hyperparameters)

Let $X_1, X_2, \ldots$ be conditionally i.i.d. given θ with common p.f. or p.d.f. $f(x \mid \theta)$. Let Ψ be a family of possible distributions over the parameter space Ω. Suppose that, no matter which prior distribution ξ we choose from Ψ, no matter how many observations $\boldsymbol{X} = (X_1, \ldots, X_n)$ we observe, and no matter what are their observed values $\boldsymbol{x} = (x_1, \ldots, x_n)$, the posterior distribution $\xi(\theta \mid \boldsymbol{x})$ is a member of Ψ. Then Ψ is called a *conjugate family* of prior distributions for samples from the distributions $f(x \mid \theta)$. It is also said that the family Ψ is *closed under sampling* from the distributions $f(\boldsymbol{x} \mid \theta)$.

Finally, if the distributions in Ψ are parametrized by further parameters, then the associated parameters for the prior distribution are called the *prior hyperparameters* and the associated parameters of the posterior distribution are called the *posterior hyperparameters*.

Sampling from a Poisson Distribution

Theorem (Gamma-Poisson model)

Suppose that $X_1, \ldots, X_n$ form a random sample from the Poisson distribution with mean $\theta > 0$, which is unknown. Suppose also that the prior distribution of θ is the gamma distribution with parameters $\alpha > 0$ and $\beta > 0$. Then the posterior distribution of θ given that $X_i = x_i, (i = 1, ..., n)$, is the gamma distribution with parameters $\alpha + \sum_{i=1}^{n} x_i$ and $\beta + n$.

Example (Gamma-Poisson model)

Suppose that customers arrive to a store at an unknown rate, θ, per hour. Assume that θ follows a gamma distribution with parameters α and β.

a) Find the posterior distribution of θ.

Sampling from a Normal Distribution

Theorem (Normal-Normal model)

Suppose that $X_1, \ldots, X_n$ form a random sample from the Normal distribution with unknown mean θ and known variance $\sigma^2 > 0$. Suppose also that the prior distribution of θ is the normal distribution with mean μ_0 and variance v_0^2. Then the posterior distribution of θ given that $X_i = x_i, (i = 1, ..., n)$, is the normal distribution with mean and variance given by

$$\mu_1 = \frac{\sigma^2\mu_0 + nv_0^2\overline{x}_n}{\sigma^2 + nv_0^2}, \qquad v_1^2 = \frac{\sigma^2 v_0^2}{\sigma^2 + nv_0^2}.$$

Sampling from a Normal Distribution

Example (Normal-Normal model)

Suppose that $X_i \mid \theta, \sigma^2 \overset{i.i.d.}{\sim} N(\theta, \sigma^2)$, with σ^2 *known*. Assume that θ follows a normal distribution with parameters μ_0 and v_0^2.

a) Find the posterior distribution of θ.

b) Show that the posterior mean can be written as

$$\mu_1 = \frac{\sigma^2}{\sigma^2 + nv_0^2}\mu_0 + \frac{nv_0^2}{\sigma^2 + nv_0^2}\overline{x}_n.$$

c) Find an expression for computing $P(\theta > 1 \mid \boldsymbol{x})$.

Sampling from a Exponential Distribution

Theorem (Gamma-Exponential model)

Suppose that $X_1, \ldots, X_n$ form a random sample from the exponential distribution with unknown parameter $\theta > 0$. Suppose also that the prior distribution of θ is the gamma distribution with parameters $\alpha > 0$ and $\beta > 0$. Then the posterior distribution of θ given that $X_i = x_i$, $(i = 1, ..., n)$, is the gamma distribution with parameters $\alpha + n$ and $\beta + \sum_{i=1}^{n} x_i$.

Example (Gamma-Exponential model)

Suppose that $X_i \mid \theta \overset{i.i.d.}{\sim} exp(\theta)$, $i = 1, \ldots, n$, $\theta > 0$, where X_i describes the lifetime of component i, and $\theta \sim Gamma(\alpha, \beta)$.

a) Find the posterior distribution of θ.

Improper Prior Distributions

- Recall that improper priors capture the idea that there is much more information in the data than is capture in our prior distribution.
- Each of the conjugate families that we have seen has an improper prior as a limiting case.

Definition

Let ξ be a nonnegative function whose domain includes the parameter space of a statistical model. Suppose that $\int_\Omega \xi(\theta)d\theta = \infty$. If we pretend as if $\xi(\theta)$ is the prior p.d.f. of θ, then we are using an improper prior for θ.

Improper Prior Distributions

Example (Improper Prior Distributions)

a) Consider a large shipment of iPhone has arrive, and the proportion of defective iPhone, θ, is unknown. A *Beta*(α, β) prior is assumed and *n* randomly selected items are inspected. What happens when $\alpha = \beta = 0$?

b) Suppose that customers arrive to a store at an unknown rate, θ, per hour. Assume that θ follows a gamma distribution with parameters α and β. What happens when $\alpha = \beta = 0$? What happens if no customers arrive in an hour?

c) Suppose that $X_i \mid \theta, \sigma^2 \overset{i.i.d.}{\sim} N(\theta, \sigma^2)$, with σ^2*known*. Assume that θ follows a normal distribution with parameters μ_0 and v_0^2. What happens if $v_0^2 = \infty$?