Examples: improper prior distributions:

a) we will consider $X_i \mid \theta \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(\theta)$, $i=1,\dots,n$

$\theta \sim \text{Beta}(\alpha, \beta)$ $\Rightarrow \xi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$ $f(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$

prior hyperparameters

so, the posterior distribution is:

$$\xi(\theta \mid x_1,\dots,x_n) \propto f_n(\underset{\sim}{x} \mid \theta)\, \xi(\theta)$$

$$\propto \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}\, \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$\propto \theta^{\alpha+\sum x_i-1}(1-\theta)^{\beta+n-\sum x_i-1}$$

so $\theta \mid \underset{\sim}{x} \sim \text{Beta}\left(\alpha+\sum_{i=1}^n x_i,\ \beta+n-\sum_{i=1}^n x_i\right)$

what happens if $\alpha=\beta=0$.

$$\xi(\theta) \propto \theta^{-1}(1-\theta)^{-1} = \frac{1}{\theta}\cdot\frac{1}{(1-\theta)} = \frac{1}{\theta}+\frac{1}{1-\theta}$$

$$\int_0^1 \xi(\theta)\,d\theta = \int_0^1 \frac{1}{\theta}+\frac{1}{1-\theta}\,d\theta = \int_0^1 \frac{1}{\theta}\,d\theta + \int_0^1 \frac{1}{1-\theta}\,d\theta$$

$$= \log(\theta)\Big|_{\theta=0}^{1} + \left[-\log(1-\theta)\right]\Big|_{\theta=0}^{1} = \infty.$$ ¡check it! ($\log = \ln$)

in this case we have that

$$\theta \mid \underset{\sim}{x} \sim \text{Beta}\left(\sum_{i=1}^n x_i,\ n-\sum_{i=1}^n x_i\right)$$

if $n=1$ and $\sum_{i=1}^n x_i = x_i = 0$, then the posterior is improper. $\int_0^1 \xi(\theta|x)\,d\theta = \infty.$

if $n>1$, $\sum_{i=1}^n x_i = 0$, then the posterior distribution

$$\xi(\theta|\underset{\sim}{x}) \propto \theta^{-1}(1-\theta)^n$$

$\int_0^1 \frac{(1-\theta)^n}{\theta}\,d\theta =$ HW. check if it is finite or not

¡ is not finite!

b) $X_i \mid \theta \sim \text{Poisson}(\theta) \quad \rightarrow \quad f(x_i \mid \theta) = \dfrac{e^{-\theta}\,\theta^{x_i}}{x_i!}$

$\theta \sim \text{Gama}(\alpha, \beta) \quad \rightarrow \quad \xi(\theta) = \dfrac{\beta^{\alpha}}{\Gamma(\alpha)}\,\theta^{\alpha-1}\,e^{-\beta\theta}$

$$\xi(\theta \mid \underline{x}) \propto e^{-n\theta}\,\theta^{\Sigma x_i}\,\theta^{\alpha-1}\,e^{-\beta\theta}$$
$$= \theta^{\alpha+\Sigma x_i - 1}\,e^{-(\beta+n)\theta}.$$

so $\quad \theta \mid \underline{x} \sim \text{Gama}\left(\alpha + \sum_{i=1}^{n} x_i\,,\ \beta+n\right)$

If $\alpha = \beta = 0$.

$\xi(\theta) \propto \dfrac{1}{\theta}$

$\int_{\Omega} \xi(\theta)\,d\theta = \int_0^{\infty} \dfrac{1}{\theta}\,d\theta = \ln\theta \Big|_0^{\infty} = \infty.$

then $\quad \theta \mid \underline{x} \sim \text{Gama}(\Sigma x_i\,,\ n)$

if nobody arrives to the store in an hour, then $\Sigma x_i = n = 0$ and the posterior distr. is improper.

c) $X_i \mid \theta, \sigma^2 \sim N(\theta, \sigma^2)$, σ^2 is known

$\theta \sim N(\mu_0, V_0^2) \longrightarrow \xi(\theta) = \dfrac{1}{\sqrt{2\pi V_0^2}} \underbrace{\exp\left\{-\frac{1}{2V_0^2}(\theta-\mu_0)^2\right\}}$

, what happens if $V_0^2 \to \infty$?

$\xi(\theta) \propto \exp\left\{-\frac{1}{2V_0^2}(\theta-\mu_0)^2\right\}.$

as $V_0^2 \to \infty$ the prior distr. of θ, $\xi(\theta)$, is a constant.

and $\quad \int_{-\infty}^{\infty} \xi(\theta)\,d\theta = \infty.$