Confidence intervals for other parameters:

$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$, μ, σ^2 are unknown.

$$\underbrace{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma'}}_{V(\underset{\sim}{X}, \mu)} \sim t_{(n-1)} , \quad \sigma' = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2}{n-1}}$$

$t_{(n-1)}$ → does not depend on any parameter.

↳ is our pivotal.

~~if~~ if $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma'} = c$ we get that $\mu = \bar{X}_n - \frac{\sigma'}{\sqrt{n}} c$.

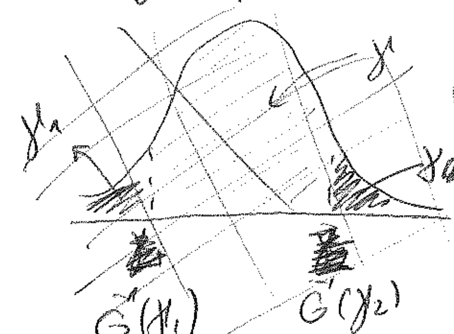
$$r(V(\underset{\sim}{X}, \mu), \underset{\sim}{X}) = \bar{X}_n - \frac{\sigma'}{\sqrt{n}} \cdot V(\underset{\sim}{X}, \mu) = \mu$$

let G be the cumulative distr. function (C.D.F) of $V(\underset{\sim}{X}, \mu)$

G^{-1} represents the quantiles of G.

let $G^{-1}(\gamma_1)$ and $G^{-1}(\gamma_2)$ be the γ_1 and γ_2 quantiles of G,

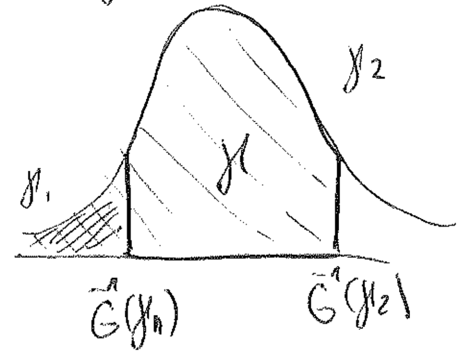
where $\gamma_2 - \gamma_1 = \gamma$.



$$P\left(G^{-1}(\gamma_1) < V(\underset{\sim}{X}, \mu) < G^{-1}(\gamma_2)\right) = \gamma$$

since $V(\underset{\sim}{X}, \mu) = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma'}$

we get the γ confidence interval



$A = \bar{X}_n - G^{-1}(\gamma_2) \cdot \frac{\sigma'}{\sqrt{n}}$ and

$B = \bar{X}_n + G^{-1}(\gamma_1) \frac{\sigma'}{\sqrt{n}}$

$\gamma_2 = \frac{1+\gamma}{2}$

Example: confidence interval for variance:

$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$.

a) $Y = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}_n}{\sigma} \right)^2$ is pivotal.

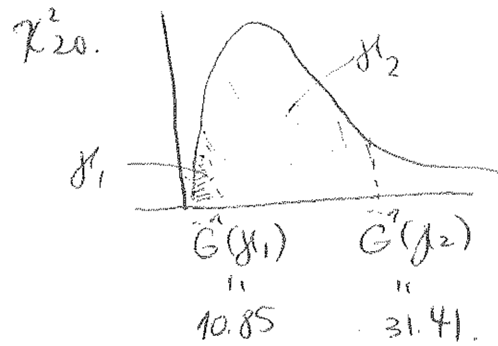
$Y \sim \chi^2_{(n-1)}$, $n = 21$.

b) $G^{-1}(\gamma_2)$, and $G^{-1}(\gamma_1)$. $\gamma_2 = 0.95$ $\gamma_1 = 0.05$.

G is the c.d.f. of a χ^2 distr. with n-1 degrees of freedom.

$G^{-1}(\gamma_1) = G^{-1}(0.05) = 10.85$

$G^{-1}(\gamma_2) = G^{-1}(0.95) = 31.41$



c) $P\left(G^{-1}(\gamma_1) < Y < G^{-1}(\gamma_2) \right) = \gamma$.

$P\left(10.85 < \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}_n}{\sigma} \right)^2 < 31.41 \right) = \gamma = 0.90$.

$P\left(10.85 < \frac{1}{\sigma^2} \sum (X_i - \bar{X}_n)^2 < 31.41 \right) = \gamma$

$\left(\begin{array}{c} a < x < b \\ \frac{1}{a} > \frac{1}{x} > \frac{1}{b} \end{array} \right)$

$P\left(\frac{1}{10.85} > \frac{\sigma^2}{\sum (X_i - \bar{X}_n)^2} > \frac{1}{31.41} \right) = \gamma$

$P\left(\frac{\sum (X_i - \bar{X}_n)^2}{31.41} < \sigma^2 < \frac{\sum (X_i - \bar{X}_n)^2}{10.85} \right) = \gamma$.

$A = \frac{\sum (X_i - \bar{X}_n)^2}{31.41}$

$B = \frac{\sum (X_i - \bar{X}_n)^2}{10.85}$.

Example: mean difference

a) X_i age of male students
Y_i age of female students

$X_i \overset{iid}{\sim} N(\mu_1, \sigma^2)$
$Y_i \overset{iid}{\sim} N(\mu_2, \sigma^2)$ > independent $i = 1, \ldots, 7$.

$$Y = \frac{\sqrt{n}\,\left[(\bar{X}_n - \bar{Y}_n) - (\mu_1 - \mu_2) \right]}{\sqrt{2}\,\sigma} \quad \text{is pivot.}$$

$X \sim N(\mu_1, \sigma^2)$
$Y \sim N(\mu_2, \sigma^2)$
$E(X-Y) = E(X) - E(Y)$
$\mu_1 - \mu_2$
$Var(X-Y) = Var(X) + Var(Y) - 2cov(X,Y)$

$\bar{X}_n \sim N(\mu_1, \sigma^2/n)$
$\bar{Y}_n \sim N(\mu_2, \sigma^2/n)$ > independent.

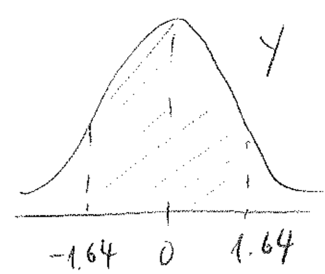
$\rightarrow \bar{X}_n - \bar{Y}_n \sim N\left(\mu_1 - \mu_2, \frac{2\sigma^2}{n}\right)$

$E(\bar{X}_n - \bar{Y}_n) = E(\bar{X}_n) - E(\bar{Y}_n) = \mu_1 - \mu_2$

$Var(\bar{X}_n - \bar{Y}_n) = Var(\bar{X}_n) + Var(\bar{Y}_n) = \frac{\sigma^2}{n} + \frac{\sigma^2}{n} = \frac{2\sigma^2}{n}$

$$\rightarrow \frac{(\bar{X}_n - \bar{Y}_n) - (\mu_1 - \mu_2)}{\sqrt{\frac{2\sigma^2}{n}}} = \frac{\sqrt{n}\,\left[(\bar{X}_n - \bar{Y}_n) - (\mu_1 - \mu_2) \right]}{\sqrt{2}\cdot\sigma} = Y$$

$Y \sim N(0,1)$



b) $G^{-1}(\gamma_1) = G^{-1}(0.05) = \Phi^{-1}(0.05) = -1.64$
$G^{-1}(\gamma_2) = \Phi^{-1}(0.95) = 1.64$

c) find A and B: $P(A < \mu_1 - \mu_2 < B) = \gamma = 0.9$.

$P\left(G^{-1}(\gamma_1) < Y < G^{-1}(\gamma_2)\right) = 0.9$

$$P\left(-1.64 < \frac{\sqrt{n}\,\left[(\bar{X}_n - \bar{Y}_n) - (\mu_1 - \mu_2) \right]}{\sqrt{2}\cdot\sigma} < 1.64\right) = 0.9$$

d) $\bar{x}_n - \bar{y}_n - 1.644 \cdot \frac{\sqrt{2} \cdot \sqrt{2}}{\sqrt{7}} = -3.41$

$\bar{x}_n + \bar{y}_n + 1.644 \frac{\sqrt{2}\sqrt{2}}{\sqrt{7}} = -0.92$

so with a 90% confidence we can say that males are younger than females in this class. The interval does not contain 0.