Bayes Estimator under square error loss:

we need to minimize $E[L(\theta,a)\mid \underset{\sim}{x}]$ with respect to a.

we are considering $L(\theta,a) = (\theta-a)^2$.

we want to minimize $E[(\theta-a)^2 \mid \underset{\sim}{x}] = \int_\Omega (\theta-a)^2 \xi(\theta\mid\underset{\sim}{x})\,d\theta$.

$$\frac{d}{da}\int_\Omega (\theta-a)^2 \xi(\theta\mid x)\,d\theta = \int_\Omega [-2(\theta-a)]\,\xi(\theta\mid\underset{\sim}{x})\,d\theta$$

$$= -2\int_\Omega \theta\,\xi(\theta\mid\underset{\sim}{x})\,d\theta + 2\int_\Omega a\,\xi(\theta\mid\underset{\sim}{x})\,d\theta$$

$$\frac{d}{da}\int_\Omega (\theta-a)^2 \xi(\theta\mid\underset{\sim}{x})\,dx = 0.$$

Then $\int_\Omega a\,\xi(\theta\mid\underset{\sim}{x})\,d\theta = \int_\Omega \theta\,\xi(\theta\mid\underset{\sim}{x})\,d\theta$.

$$a\underbrace{\int_\Omega \xi(\theta\mid\underset{\sim}{x})\,d\theta}_{1} = \underbrace{\int_\Omega \theta\,\xi(\theta\mid x)\,d\theta}_{E(\theta\mid\underset{\sim}{X})}$$

so $a = E(\theta\mid\underset{\sim}{X})$.

now we ~~that~~ ~~de~~ check that a is a minimum.

$$\frac{d^2}{da^2}\int_\Omega (\theta-a)^2\xi(\theta\mid x)\,d\theta = \frac{d}{da}\left[-2\int_\Omega \theta\,\xi(\theta\mid\underset{\sim}{x})\,d\theta + 2\int_\Omega a\,\xi(\theta\mid x)\,d\theta\right]$$

$$= 0 + 2\underbrace{\int_\Omega \xi(\theta\mid x)\,d\theta}$$

$$= 2\cdot 1$$

$$= 2 > 0.$$

Therefore $a = E(\theta\mid\underset{\sim}{X})$ is a minimum of $E[(\theta-a)^2\mid x]$

and Bayes estimator is

$$\delta^* = \delta^*(X_1,\ldots X_n) = E(\theta\mid X_1,\ldots X_n) = E(\theta\mid\underset{\sim}{X}).$$

Example:

$X_i \mid \theta \overset{iid}{\sim} \text{Bernoulli}(\theta)$

$\theta \sim \text{Beta}(a, b)$

show that under square error loss $\delta^*(\underset{\sim}{X}) = \dfrac{(a + \sum_{i=1}^{n} X_i)}{a+b+n}$

$$\xi(\theta \mid \underset{\sim}{x}) \propto \theta^{\sum x_i} (1-\theta)^{n - \sum x_i} \theta^{a-1} (1-\theta)^{b-1}$$

$$= \underbrace{\theta^{a + \sum x_i - 1} (1-\theta)^{b + n - \sum x_i - 1}}_{\text{Beta}(a + \sum x_i,\ b + n - \sum x_i)}$$

Since we have square error loss, we know that Bayes estimator $\delta^*(\underset{\sim}{X}) = E(\theta \mid \underset{\sim}{X})$. And since $\theta \mid \underset{\sim}{x} \sim \text{Beta}(a + \sum x_i, b + n - \sum x_i)$ we have that

$$\delta^*(\underset{\sim}{X}) = \frac{a + \sum_{i=1}^{n} X_i}{a+b+n}$$

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other loss ~~fun~~ function that can be considered is

$$L(\theta, a) = \begin{cases} 3(\theta - a) & \theta \geq a \\ (\theta - a) & \theta < a \end{cases}$$

for finding $\delta^*(\underset{\sim}{x})$ we need to minimize, w.r.t. a

$$\int_{\Omega} L(\theta, a)\xi(\theta \mid x) d\theta = \int_a^{\infty} 3(\theta - a)\xi(\theta \mid \underset{\sim}{x})\, d\theta + \int_{-\infty}^{a} (\theta - a)\xi(\theta \mid \underset{\sim}{x})\, d\theta.$$

Bayes estimator for large samples.

From the last example we have that

$$\theta|\underset{\sim}{x} \sim Beta(a+\Sigma x_i, b+n-\Sigma x_i)$$, and

$$\delta^*(\underset{\sim}{X}) = E(\theta|\underset{\sim}{X}) = \frac{a+\Sigma x_i}{b+a+n}.$$

i) considering $a=b=1$: for the prior: $E(\theta) = 1/2 = 0.5$

for Bayes estimate.

$$\delta^* = \delta^*(\underset{\sim}{x}) = E(\theta|\underset{\sim}{x}) = \frac{1+10}{1+1+100} = \frac{11}{102} = 0.1078.$$

ii) considering $a=1$, $b=2$; for the prior $E(\theta) = \frac{1}{3} = 0.\overline{33}$

for Bayes estimate

$$\delta^* = \delta^*(\underset{\sim}{x}) = E(\theta|\underset{\sim}{x}) = \frac{1+10}{2+1+100} = \frac{11}{103} = 0.1067.$$

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consistency:

First note that

$$\delta^*(\underset{\sim}{X}) = \frac{a+\Sigma X_i}{a+b+n} = \underbrace{\frac{a+b}{a+b+n}}_{\substack{\to 0 \\ n\to\infty}} \cdot \underbrace{\frac{a}{a+b}}_{E(\theta)} + \underbrace{\frac{n}{a+b+n}}_{\substack{\to 1 \\ n\to\infty}} \cdot \overline{X_n}$$

we have that $\delta^*(\underset{\sim}{X})$ converges in probability to θ, as $n\to\infty$.

(because $\overline{X_n}$ converges in prob. to θ)